Factor-Biased Public Capital and Private Capital Crowding Out*

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Abstract

This paper studies the dynamic effects of public investment on private capital accumulation in a general equilibrium macroeconomic model of a small open economy with factor-biased public capital. I show that public investment induces rather complex private capital dynamics—falling in the short and in the long run, but potentially increasing along transition—if public capital augments private capital and private inputs are gross complements in production. Whether private investment is crowded in or out during transition critically depends on parameters that empirically hard to measure, such as the labor supply elasticity and the elasticity of substitution between private inputs—a small increase in the latter from 0.5 to 0.6, for instance, would turn a totally negative transitional effect into a predominantly positive one. These results help rationalize the lack of empirical consensus on the relationship between public and private investment.

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1 Introduction

The recent global economic and financial crisis—to which governments of many advanced countries reacted with fiscal stimulus measures based on public capital expenditures—sparked a renewed interest in the macroeconomic effects of public investment.[1] A central issue for policymakers is how public capital spending affects private capital formation. The empirical evidence on the public-private investment relationship is rather mixed, however.[2] This paper studies the dynamic effects of public investment on private capital formation in a general equilibrium model of small open economy. Whereas the crowding-in/out effects of public investment have typically been rationalized in terms of the degree of substitution between public and private capital in production, this paper assigns a central role to the dynamic reaction of labor employment and its complementarity with private capital. In particular, I show how the dynamic interplay between private capital and labor can generate transitional crowding-in effects even when public investment crowds out public capital formation both in the short and in long run.

From a supply-side perspective, the relationship between public capital spending and private capital formation depends on the specification of the production technology. First, public capital can enter the production function in a factor-neutral way (in which case the relative marginal productivities of private capital and labor is unaffected) or in a factor-biased way (in which case it increases the marginal productivity of one factor relative to the other). Second, private capital and labor can be specified to be gross complements (i.e., with an elasticity of substitution below one) or gross substitutes (i.e., with an elasticity of substitution above one) in production. Most studies employ a Cobb-Douglas production specification, which constrains the elasticity of substitution between private capital and labor to unity and, consequently, forces public capital to be factor-neutral. Chirinko (2008)

[1]Some examples include Leeper, Walker, and Yang (2010); Leduc and Wilson (2012); Bom and Ligthart (2014a); Bouakez, Guillard, and Rolleau-Pasdeloup (2014); Abiad, Furceri, and Topalova (2015); and Clancy, Jacquinot, and Lozej (2016).

[2]Voss (2002) finds evidence of crowding out of private investment by public investment shocks in Canada and the United States. Perotti (2004) finds similar results also for Australia, Germany, and the United Kingdom. Afonso and St. Aubyn (2009) document both crowding-in effects of public investment in eight out of 17 developed economies, and crowding-out effects in the remaining nine. The evidence is also mixed for developing economies. Cavallo and Daude (2011) find mostly crowding-out effects in a sample of 116 countries, whereas Eden and Kraay (2014) report large positive effects of public investment on private investment in a panel of 39 countries.

observes, however, that the bulk of empirical evidence suggests an elasticity somewhere in the range 0.4–0.6. To depart from unitary elasticities, I adopt a constant elasticity of substitution (CES) production function and focus on values within the range 0.4–0.6. I consider both capital- and labor-augmenting public capital technologies, but focus on the former as a baseline scenario.

I embed the CES production function into a dynamic general equilibrium model of small open economy facing a perfectly elastic supply of capital from abroad at the exogenously-given rate of interest. Perfectly competitive firms choose private investment and labor taking the stock of public capital as given. To prevent instantaneous adjustment of physical capital stocks, I impose adjustments costs on private capital formation. This feature gives rise to a responsive Tobin's q, which temporarily absorbs shocks to the marginal productivity of private capital. I assume the economy borrows/lends in world capital markets under full commitment and abstract from repayment default and risk premium issues. To keep the public sector as simple as possible, I assume the government finances public capital by means of a lump-sum tax. This assumption rules out potential distortions from other tax instruments and allows us to focus directly on the productive role of public capital.

The household side of the economy consists of infinitely-many overlapping and disconnected generations of finitely-lived agents with a leisure-labor choice.The production and household sectors of the model are therefore connected through endogenous labor supply. I employ the overlapping-generations structure as a convenient technical device to obtain an endogenously-determined steady state, which avoids the unit-root property of small open economy models with fixed interest rates (see Schmitt-Grohé and Uríbe, 2003). I adopt Blanchard's (1985) perpetual-youth formulation, in which households face a constant probability of death independently of age.[3] Although this assumption oversimplifies mortality risk and rules out life-cycle behavior, it captures the intergenerational disconnectedness—i.e., lack of perfect intergenerational altruism—observed in the data (see, e.g., Barczyk, 2016) in relatively simple way. Moreover, it encompasses the common infinitely-lived representative agent framework as a special case.

I first study analytically the long-run effects of a permanent public investment increase

[3] Heijdra and Meijdam (2002) employ a similar same household structure to study the intergenerational welfare effects of public investment, but assume exogenous labor supply.

on private capital. I show that (private capital-augmenting) public capital crowds out private capital in the long run if (and only if) the elasticity of substitution is sufficiently small and the output elasticity of public capital is sufficiently large. Long-run crowding out occurs because, in this case, public capital decreases the marginal productivity of private capital in the long run. The labor supply elasticity plays a minor role in the long run, affecting the private capital effect quantitatively through the labor employment effect, but without affecting the qualitative crowding-out result.

I then turn to the impact and transitional dynamics of a public investment shock. I log-linearize the model around an initial steady state and solve analytically for the impulse response functions. I calibrate the model with empirically-plausible parameter values and focus on the quantitative and qualitative features of the numerical simulations. The most important result of this exercise is that a capital-augmenting increase in public investment causes rather complex dynamic effects to private capital formation. It falls in the short run and in the long run, but may increase along transition between steady states if labor supply is relatively elastic. Moreover, the transitional response of private capital is very sensitive, both quantitatively and qualitatively, to small changes in the output elasticity of public capital and in elasticity of substitution between private capital and labor. For instance, an increase in the elasticity of substitution from 0.5 to 0.6 suffices to turn the private capital response from totally negative to predominantly positive.

This paper relates to several studies on the real macroeconomic effects of public investment. Fisher and Turnovsky (1998) also study explicitly the effects of public investment on private capital accumulation, but in a model of exogenous labor. In their model, crowding out arises as the result of congestion effects, especially for a low elasticity of substitution between private and public capital. Baxter and King (1993) and Turnovsky and Fisher (1995) study the macroeconomic effects of temporary and permanent shocks to government spending, including its investment component, using a similar neoclassical model of a closed economy. Leeper, Walker, and Yang (2010) study the role of public investment as a countercyclical fiscal policy tool, focusing on implementation delays. Bom and Ligthart (2014a) investigate how distortionary labor income taxes constrain the effects of public investment under balanced-budget rules. None of these papers, however, consider factor-biased public capital and the role of the elasticity of substitution between private capital and labor.

The rest of the paper is structured as follows. Section 2 describes the main elements of model. Section 3 derives analytically the steady-state effects of public investment, specifying the conditions under which public capital crowds out private capital in the long run. Section 4 studies the transitional dynamics of a public investment impulse, discussing in turn the solution of the log-linearized model, the calibration strategy, and the simulation results. Section 5 concludes the paper.

2 The Model

This section describes the dynamic general equilibrium model. The model consists of a production sector, a household sector, a public sector, and a foreign sector. The economy is small and open, so that the interest rate is exogenously determined in world capital markets. Capital is perfectly mobile across borders. The model is specified in real terms. Time is continuous. I first discuss the production technology and the behavior of firms in Section 2.1. Section 2.2 turns to the behavior of individual households and the aggregate household sector. Finally, Section 2.3 describes the public sector, the foreign sector, and market equilibrium.

2.1 Firms

The production sector of the economy consists of infinitely-many identical firms producing a homogeneous good under perfect competition in the output and factor markets. Firms have access to a production technology that transforms private capital, $K(t)$, and labor, $L(t)$, into output, $Y(t)$. Firms choose $K(t)$ and $L(t)$ taking as given the existing stock of public capital, $K_G(t)$, which is provided by the government. I allow for flexible substitutability between private capital and labor by assuming a constant elasticity of substitution (CES) production specification:

$$Y(t) = \left\{ [E_K(t)K(t)]^{\frac{\sigma_Y - 1}{\sigma_Y}} + [E_L(t)L(t)]^{\frac{\sigma_Y - 1}{\sigma_Y}} \right\}^{\frac{\sigma_Y}{\sigma_Y - 1}}, \tag{1}$$

where $E_K(t)$ and $E_L(t)$ are capital- and labor-augmenting technical change terms (discussed below). Denoting the marginal productivities of private capital and labor by $Y_K(t)$ and $Y_L(t)$, the elasticity of substitution between the two factors is $\sigma_Y \equiv \frac{d \ln[L(t)/K(t)]}{d \ln[Y_K(t)/Y_L(t)]} > 0$.[4] Private capital and labor are said to be gross complements if $\sigma_Y < 1$, and gross substitutes if $\sigma_Y > 1$. The case $\sigma_Y = 1$ corresponds to the Cobb-Douglas production function. Chirinko (2008) argues that the bulk of the empirical evidence suggests a value for σ_Y in the range 0.4–0.6. As a baseline, therefore, I assume gross complementarity between private capital and labor.

Apart from private capital and labor, private sector output depends also on public capital services, which are assumed to be proportional to the *stock* of public capital. In this paper, public capital is defined as core infrastructure that is directly productive to private firms—e.g., roads and highways, water systems, energy utilities, etc. Public capital is modeled as a pure public good, provided by the government free of charge and not subject to congestion.[5] Public capital, $K_G(t)$, enters the production function through the technical change terms $E_K(t)$ and $E_L(t)$ in a factor-augmenting fashion:

$$E_i(t) \equiv \rho_i K_G(t)^{\eta_i}, \qquad i = \{K, L\}, \tag{2}$$

where $\eta_i \geq 0$ represents the elasticity of the factor-augmentation term with respect to public capital, and $\rho_i > 0$ is a scaling factor. If $\eta_K = \eta_L$, public capital is factor-neutral—i.e., it augments private inputs proportionally. In this paper, however, I allow for factor-biased public capital, which requires $\eta_K \neq \eta_L$. Focusing only on pure factor-augmentation cases, private capital-augmenting public capital is captured by $\eta_K > \eta_L = 0$, whereas labor-augmenting public capital is described by $\eta_L > \eta_K = 0$. In light of its interpretation as core infrastructure, it seems natural to assume public capital to be private capital-augmenting. The labor-augmenting specification may be more relevant for other, less directly productive, forms of public spending—such as education or research grants. Hence, the baseline public capital specification assumes $\eta_K > \eta_L = 0$.

[4] The CES production function embeds the Leontief technology for $\sigma_Y = 0$, and the linear technology for $\sigma_Y \to \infty$. I do not consider these limiting cases in this paper.

[5] Fisher and Turnovsky (1998) and Dioikitopoulos and Kalyvitis (2008) explicitly focus on the effects of public capital congestion.

To understand the importance of the elasticity of substitution between private capital and labor in connection with the type of factor-augmentation induced by public capital, consider the ratio of marginal productivities of private capital and labor:

$$\frac{Y_K(t)}{Y_L(t)} = \left(\frac{\rho_K}{\rho_L}\right)^{\frac{\sigma_Y - 1}{\sigma_Y}} K_G(t)^{\frac{(\eta_K - \eta_L)(\sigma_Y - 1)}{\sigma_Y}} \left[\frac{K(t)}{L(t)}\right]^{-\frac{1}{\sigma_Y}}, \tag{3}$$

If public capital is purely capital-augmenting (i.e., $\eta_K > \eta_L = 0$) and private inputs are gross substitutes (i.e., $\sigma_Y > 1$), an increase in $K_G(t)$ raises the marginal productivity of capital relative to that of labor. Hence, public capital is biased toward private capital. In the more empirically-plausible case of gross complementarity (i.e., $\sigma_Y < 1$), however, an increasing in capital-augmenting $K_G(t)$ lowers the relative marginal productivity of private capital. Public capital is then biased towards labor.

The production specification (1) features constant returns to scale across private inputs. Hence, denoting the output elasticity of private factor $j = \{K, L\}$ by $\theta_j(t) \equiv Y_j(t)j(t)/Y(t)$, it holds that $\theta_K(t) + \theta_L(t) = 1$. Moreover, because public capital generates a production externality, the production technology exhibits increasing returns to scale across all inputs—i.e., $\theta_K(t) + \theta_L(t) + \theta_G(t) \geq 1$, where $\theta_G(t) \equiv \frac{\partial Y(t)}{\partial K_G(t)} \frac{K_G(t)}{Y(t)} \geq 0$ denotes the output elasticity of public capital. Note that this elasticity can be written as $\theta_G(t) = \theta_K(t)\eta_K + \theta_L(t)\eta_L$, a convex combination of η_K and η_L. Because I focus on pure factor-augmentation cases, $\theta_G(t) = \theta_K(t)\eta_K$ if public capital augments private capital, and $\theta_G(t) = \theta_L(t)\eta_L$ if public capital augments labor. I rule out endogenous growth by requiring diminishing returns to reproducible factors (i.e., private and public capital), which amounts to imposing $\eta_K + \theta_K(t) < 1$; this condition is easily met for plausible parameter values (see Section 4.2).

I postulate convex adjustment costs to private capital. Capital adjustment costs are not only empirically relevant but also technically convenient in generating nontrivial transitional dynamics in private capital in the case of small open economy facing an exogenously-given interest rate.[6] In particular, I assume that net capital formation relates to gross

[6] Cooper and Haltiwanger (2006) find that, although a combination of convex and non-convex adjustment costs is necessary to fit the data at the plant level, convex adjustment costs fit the aggregate reasonably well.

investment, $I(t)$, according to

$$\dot{K}(t) = \left[\Phi\left(\frac{I(t)}{K(t)}\right) - \delta\right] K(t), \tag{4}$$

where δ is the depreciation rate of private capital and $\Phi(\cdot)$ is a strictly concave installation cost function of private capital (i.e., $\Phi'(\cdot) > 0$ and $\Phi''(\cdot) < 0$) featuring zero net capital formation and adjustment costs at the origin (i.e., $\Phi(0) = 0$ and $\Phi'(0) = 1$).

The representative firm maximizes the net present value of its cash flow:

$$V(t) \equiv \int_t^\infty \left[Y(\tau) - w(\tau)L(\tau) - I(\tau)\right] e^{r(t-\tau)} d\tau, \tag{5}$$

given the production technology (1) and the existing stock of public capital, $K_G(t)$, and subject to the capital accumulation constraint (4). In (5), $w(t)$ and r denote the gross wage rate and the exogenously-given interest rate. Note, moreover, that the price of output and investment goods are both normalized to unity. The co-state variable of the firm's optimization problem, denoted by $q(t)$, corresponds to Tobin's q—i.e., the market value of installed capital relative to replacement cost—and is governed by

$$\dot{q}(t) = -q(t)\left[\Phi\left(\frac{I(t)}{K(t)}\right) - \frac{I(t)}{K(t)}\Phi'\left(\frac{I(t)}{K(t)}\right) - (r+\delta)\right] - Y_K(t). \tag{6}$$

where $\dot{q}(t) \equiv dq(t)/dt$ (a notational convention I use for all time derivatives in this paper). Because capital adjustment costs prevent the capital stock from adjusting instantaneously, shocks to its marginal productivity (given by the the last term) are temporarily absorbed by Tobin's q. The static first-order conditions are:

$$w(t) = Y_L(t), \tag{7}$$

$$1 = q(t)\Phi'\left(\frac{I(t)}{K(t)}\right). \tag{8}$$

Equation (7) is a standard labor demand function setting the wage rate to the marginal productivity of labor, whereas (8) pins down the optimal investment level conditional on the existing stock of capital and Tobin's q. Note that (8) boils down to $q(t) = 1$ in a model without adjustment costs, in which case $\dot{q}(t) = 0$ implies that (6) reduces to the standard condition $Y_K(t) = r + \delta$.

2.2 Households

Following Heijdra and Meijdam (2002) and Bom and Ligthart (2014a), I assume overlapping generations of finitely-lived households. Households face a constant instantaneous probability of death measured by β, the same rate at which new households are born. The population size is thus constant and can be normalized to one. Because households do not have a bequest motive, generations are disconnected. Households insure against mortality risk by contracting actuarially-fair 'reverse' life insurance, which adds an extra return on financial wealth equal to the probability of death, β (cf. Blanchard, 1985). Individual households optimally decide on private consumption spending, $C(v,t)$, and on the split of one unit of time between labor, $L(v,t)$, and leisure, $1 - L(v,t)$. At time t, a household born at $v \leq t$ maximizes

$$\Lambda(v,t) \equiv \int_t^\infty \{\varepsilon_C \ln C(v,t) + (1-\varepsilon_C)\ln[1-L(v,t)]\}\, e^{(\alpha+\beta)(t-\tau)} d\tau, \tag{9}$$

subject to the flow budget constraint

$$\dot{A}(v,t) = (r+\beta)A(v,t) + w(t) - T(t) - X(v,t). \tag{10}$$

In the objective function (9), α and ε_C denote the pure rate of time preference and the consumption weight in instantaneous utility. In the constraint (10), $A(v,t)$ denotes the real stock of financial wealth, $w(t)$ is the (age-independent) real wage, $T(t)$ are lump-sum taxes, and $X(v,t) \equiv w(t)\,[1-L(v,t)] + C(v,t)$ denotes 'full' consumption—i.e., the combined market value of consumption and leisure. I assume taxes cannot exhaust labor income (i.e., $T(t) < w(t)L(t)$), so that consumption and saving can both be positive for an individual with no financial assets (as is the case at birth). Note the presence of β as an extra component in the discount factor (reflecting the mortality risk parameterized by β), and also as an extra return on financial wealth (representing the insurance premium on that risk). The standard infinitely-lived representative agent framework obtains for $\beta = 0$.

Solving the household's problem by two-stage budgeting gives, in the first stage, the

Euler equation for individual full consumption

$$\frac{\dot{X}(v,t)}{X(v,t)} = r - \alpha, \tag{11}$$

which governs the intertemporal allocation of full consumption. The second stage gives first-order conditions for consumption and leisure demand:

$$C(v,t) = \varepsilon_C X(v,t), \tag{12}$$

$$w(t)\,[1 - L(v,t)] = (1 - \varepsilon_C) X(v,t), \tag{13}$$

which determine the intratemporal allocation of full consumption across private consumption and leisure.

To aggregate variables at the individual level, note that the size of each living cohort v is a fraction $\beta e^{\beta(v-t)}$ of total population. Hence, aggregating a generic individual variable $x(t,v)$ amounts to finding $x(t) = \int_{-\infty}^{t} x(v,t)\beta e^{\beta(v-t)}dv$. For financial wealth, aggregating $A(v,t)$ and taking its time derivative delivers the aggregate version of the flow budget constraint (10):

$$\dot{A}(t) = rA(t) + w(t) - T(t) - X(t). \tag{14}$$

Notice that the extra return component β in (10) washes out in the aggregate, as it merely represents transfers of financial wealth between generations. Following the same procedure for full consumption gives the aggregate Euler equation

$$\frac{\dot{X}(t)}{X(t)} = r - \alpha - \frac{\beta(\alpha+\beta)A(t)}{X(t)}, \tag{15}$$

which augments (11) with a 'generational turnover' effect (represented by the last term). Thus, an economy with positive aggregate financial wealth (i.e., $A(t) > 0$) in steady state (i.e., with $\dot{X}(t) = 0$) implies $r > \alpha$, which in turn yields rising consumption profiles at the individual level. Equations (12) and (13) keep the same form in the aggregate: $C(t) = \varepsilon_C X(t)$ and $w(t)\,[1 - L(t)] = (1 - \varepsilon_C)X(t)$.

2.3 Public Sector, Foreign Sector, and Market Equilibrium

The government spends on public capital investment, $I_G(t)$, and public consumption, $C_G(t)$. The flow of government spending is entirely financed by lump-sum taxes, $T(t)$, so that the government must satisfy the budget constraint

$$T(t) = I_G(t) + C_G(t). \tag{16}$$

To focus directly on the productivity spillovers of public capital, I abstract from spillover effects on the consumption side and also from tax distortions that would potentially arise under alternative tax instruments.[7] Government capital accumulates according to

$$\dot{K}_G(t) = \left[\Phi_G\left(\frac{I_G(t)}{K_G(t)}\right) - \delta_G\right] K_G(t), \tag{17}$$

where $\Phi_G(\cdot)$ is a strictly concave installation function of public capital—satisfying the same properties as $\Phi(\cdot)$ in (4)—and δ_G is the rate of depreciation of public capital.

Financial capital moves unrestrictedly across borders. Denoting net exports by $Z(t)$, net foreign assets, $F(t)$, evolves according to $\dot{F}(t) = rF(t) + Z(t)$. I assume that the two available assets—i.e., shares of domestic firms and the international bond—are perfect substitutes in the household's portfolio. Hence, financial equilibrium amounts to $A(t) = V(t) + F(t)$, where $V(t) \equiv q(t)K(t)$ denotes the stock market value of firms. I abstract from nominal and real rigidities in the labor and goods markets. The goods market thus clears at all instants of time: $Y(t) = C(t) + C_G(t) + I(t) + I_G(t) + Z(t)$.

3 Long-Run Effects of Public Investment

Section 4.1 below argues that the model is characterized by a unique and locally-stable steady state. Before discussing the dynamics of the model, this section studies analytically the marginal long-run level effects of a public investment increase ($dI_G > 0$). Because the analysis amounts to comparing pre- and post-shock steady states, I drop time indices and denote variables in the initial by the subscript 0 and in the new steady state by the

[7] Bom and Ligthart (2014a) consider the case of proportional labor income taxes.

subscript 1. For intuition, I complement the analytical results with a graphical illustration of the long-run forces at work (see Figure 1).

3.1 Full Consumption and Labor

Let us start by the long-run effect of public investment on full consumption. Consider the consumption-saving subsystem defined by equations (14) and (15) in steady state—i.e., with $\dot{A}(t) = 0$ and $\dot{X}(t) = 0$ imposed. Dropping time indices and solving for X, the resulting equations are:

$$X = rA + w - T, \tag{18}$$

$$X = \frac{\beta(\alpha+\beta)}{r-\alpha}A = \frac{r\omega_X}{\omega_A}A, \tag{19}$$

where $\omega_X \equiv X_0/Y_0$ and $\omega_A \equiv rA_0/Y_0$ are steady-state shares. Steady-state conditions (18) and (19) can be depicted as straight, positively-sloped lines in the X–A space, conditional on w and T (see Panel (a) of Figure 1). Denote these lines by $\dot{A} = 0$ and $\dot{X} = 0$, respectively. The $\dot{X} = 0$ line is determined solely by fixed parameters and is thus unaffected by public investment. The position of the $\dot{A} = 0$ line, however depends on w and T, which are affected by public investment; an increase in w shifts $\dot{A} = 0$ up, while an increase in T shifts it down. For $w = w_0$ and $T = T_0$, its initially located at $\dot{A} = 0|_0$. Note that w is determined in the labor market (discussed below), which is depicted in Panel (b); the two panels are therefore connected.

Solving the steady-state conditions $\dot{A} = 0$ and $\dot{X} = 0$ leads to an expression for full consumption as linear function of $w - T$. Differentiating this function with respect to I_G gives rise to one equation of a linear system of long-run multipliers (see Appendix A.1). By solving this system for dX/dI_G one arrives at the reduced-form multiplier

$$\frac{dX}{dI_G} = \frac{\omega_X}{\omega_G^I}\left[\frac{\theta_G(1+\omega_{LL}) - \omega_G^I}{\theta_L(1+\omega_{LL}) - \omega_T}\right], \tag{20}$$

where $\omega_{LL} \equiv L_0/(1-L_0)$ is the leisure-labor ratio (and also the Frisch elasticity of labor supply), $\theta_L \equiv w_0L_0/Y_0$ is the labor share of total income, $\theta_G \equiv Y_{G0}K_{G0}/Y_0$ is the output elasticity of public capital, $\omega_G^I \equiv I_{G0}/Y_0$ is the public investment-to-GDP ratio, and $\omega_T \equiv$

T_0/Y_0 is the tax revenues-to-GDP ratio. Noting that $(1+\omega_{LL})\theta_L-\omega_T=\omega_X-\omega_A>0$, public investment raises full consumption in the long run if and only if $dw/dI_G=\theta_G(1+\omega_{LL})/\omega_G^I>1$—i.e., if and only if the long-run increase in wages more than compensates the long-run tax raise needed to finance the public investment increment. This case is represented in Panel (a) of Figure 1 by an upward shift of the $\dot{A}=0|_0$ line.

Let us turn now to the labor market, which is described by labor demand and labor supply:

$$w = E_L^{\frac{\sigma_Y-1}{\sigma_Y}}\left(\frac{Y}{L}\right)^{\frac{1}{\sigma_Y}}, \tag{21}$$

$$w = (1-\varepsilon_C)\frac{X}{1-L}, \tag{22}$$

where the former corresponds to (7) and the latter obtains from aggregating (13). Panel (b) of Figure 1 depicts the two curves in the w–L space as L^d and L^s, respectively. The L^d curve slopes downward and stands initially at L_0^d, whereas the L^s curve slopes upward and is initially at L_0^s. An increase in public investment—and thus in the stock of public capital— affects the labor market directly through E_L, and indirectly through X and K (which is embedded in Y). That X affects L reflects the wealth effect in labor supply and connects the labor market with the consumption-saving subsystem depicted in Panel (a). The dependency of L on K follows from factor complementarity in production and links the labor market with the capital market represented in Panel (c) (discussed below).

Equations (21) and (21) implicitly define the equilibrium level of labor employment as $L=L(K,K_G,X)$. By fully differentiating this function with respect to I_G and solving for dL/dI_G (see Appendix A.1) one finds

$$\frac{dL}{dI_G}=\frac{\omega_{LL}}{y_K\theta_L\omega_G^I}\left[\frac{\theta_L\omega_G^I-\theta_G\omega_T}{\theta_L(1+\omega_{LL})-\omega_T}\right], \tag{23}$$

where $y_K\equiv Y_0/K_0$ is the average productivity of capital in the initial steady state. Note that the long-run labor multiplier depends neither on the elasticity of substitution nor on the particular type of factor augmentation, only on the overall output elasticity of public capital. Proposition 1 summarizes the signs of the long-run effects of public investment on full consumption and labor.

Proposition 1 (Signs of Full Consumption and Labor Multipliers)**.** *Define* $\underline{\theta_G} \equiv \frac{\omega_G^I}{1+\omega_{LL}} \geq 0$ *and* $\overline{\theta_G} \equiv \frac{\theta_L \omega_G^I}{\omega_T} \geq 0$*. Then,* $\underline{\theta_G} < \overline{\theta_G}$*, and the long-run multipliers of full consumption and labor, given by (20) and (23), can be signed as follows:*

$$\begin{cases} \frac{dX(\infty)}{dI_G} \leq 0 \quad \textit{and} \quad \frac{dL(\infty)}{dI_G} > 0, \quad \textit{if} \quad \theta_G \leq \underline{\theta_G}, \\ \frac{dX(\infty)}{dI_G} > 0 \quad \textit{and} \quad \frac{dL(\infty)}{dI_G} \geq 0, \quad \textit{if} \quad \underline{\theta_G} < \theta_G \leq \overline{\theta_G}, \\ \frac{dX(\infty)}{dI_G} > 0 \quad \textit{and} \quad \frac{dL(\infty)}{dI_G} < 0, \quad \textit{if} \quad \theta_G > \overline{\theta_G}. \end{cases}$$

Proof. That $\frac{dX(\infty)}{dI_G} \lesseqgtr 0$ if and only if $\theta_G \lesseqgtr \frac{\omega_G^I}{1+\omega_{LL}} \equiv \underline{\theta_G}$ and $\frac{dL(\infty)}{dI_G} \gtreqless 0$ if and only if $\theta_G \lesseqgtr \frac{\theta_L \omega_G^I}{\omega_T} \equiv \overline{\theta_G}$ follows directly from the numerators of the bracketed fractions of (20) and (23), after noting that $\theta_L(1+\omega_{LL}) > \omega_T$. This last inequality can also be rearranged as $\frac{\omega_G^I}{1+\omega_{LL}} < \frac{\theta_L \omega_G^I}{\omega_T}$, so that $\underline{\theta_G} < \overline{\theta_G}$. □

For a graphical interpretation of Proposition 1, notice that the public capital externality is split into small, moderate, and large values. For values of θ_G in the moderate range $(\underline{\theta_G}, \overline{\theta_G})$—argued below to be the empirically relevant case (see Section 4.1)—L_0^d moves to L_1^d in Panel (a) of Figure 1. In Panel (b), wages increase by more than taxes, so that $\dot{A} = 0|_0$ line shifts up to $\dot{A} = 0|_1$, which in turn triggers a wealth effect that shifts L_0^s leftward to L_1^s. As a result, the long-run full consumption and labor effects are both positive. For small values of θ_G, L_0^d shifts only to L_S^d, and the $\dot{A} = 0|_0$ schedule goes down; because the wealth effect is now negative, the L_0^s curve moves to the right. The full consumption multiplier is then negative, but the labor multiplier is even more strongly positive. Finally, for large values of θ_G, labor demand shifts to L_L^d and the $\dot{A} = 0|_0$ line jumps to $\dot{A} = 0|_L$, causing a strong wealth effect on labor supply to L_L^s. The long-run effect on full consumption is then positive but the labor multiplier is negative.

3.2 Long-Run Crowding Out

Let us now turn to the long-run effects of public investment on private capital. Steady-state equilibrium in the capital market is determined by conditions $\dot{K}(t) = 0$ in (4) and

$\dot{q}(t) = 0$ in (8):

$$q = \frac{1}{\Phi'\left(\Phi^{-1}(\delta)\right)} = q_0, \tag{24}$$

$$q = \frac{1}{r}\left[E_K^{\frac{\sigma_Y - 1}{\sigma_Y}}\left(\frac{Y}{K}\right)^{\frac{1}{\sigma_Y}} - \frac{I}{K}\right], \tag{25}$$

Panel (c) of Figure 1 depicts the two conditions in the q–K space as $\dot{K} = 0$ and $\dot{q} = 0$, respectively. The former is represented by an horizontal line at the unique steady-state value of q_0, whereas the latter gives rise to a downward-sloping curve. The capital market is initially at E_0, the crossing point of $\dot{q} = 0|_0$ and $\dot{K} = 0$, with an equilibrium stock of capital of $K = K_0$. Public investment affects long-run private capital directly through E_K and indirectly through L (which is embedded in Y), both of which affect the $\dot{q} = 0$ curve.

As derived in Appendix A.1, the reduced-form private capital multiplier reads

$$\frac{dK}{dI_G} = \frac{1}{y_K \theta_L \omega_G^I}\left\{\theta_G - \eta_K(1 - \sigma_Y) + \omega_{LL}\left[\frac{\theta_L \omega_G^I - \theta_G \omega_T}{\theta_L(1 + \omega_{LL}) - \omega_T}\right]\right\}. \tag{26}$$

The combined first two terms within brackets (i.e., $\theta_G - \eta_K(1 - \sigma_Y)$) capture the direct effect public capital on private capital productivity, whereas the last term captures the indirect productivity effect of public capital through changes in labor employment. While the indirect labor effect has the same sign as the labor multiplier (determined in Proposition 1), the sign of the direct productivity effect critically depends on the technology parameters θ_G and η_K (i.e., size and factor-augmentation type of the public capital spillover), and σ_Y (i.e., elasticity of substitution between private capital and labor). In particular, the private capital multiplier can be negative for relatively large values of η_K and small values of σ_Y. Proposition 2 gives the necessary and sufficient conditions under which public investment crowds out private capital in the long run.

Proposition 2 (Long-Run Crowding Out)**.** *Define the threshold values*

$$\underline{\sigma_Y} \equiv \theta_L - \frac{\omega_{LL}}{\eta_K}\left[\frac{\theta_L \omega_G^I - \theta_G \omega_T}{\theta_L(1 + \omega_{LL}) - \omega_T}\right] \lesseqgtr 0,$$

$$\underline{\eta_K} \equiv \frac{\omega_{LL}\theta_L \omega_G^I}{\omega_{LL}\theta_K \omega_T + \theta_L[\theta_L(1 + \omega_{LL}) - \omega_T]} \geq 0.$$

Then, the long-run multiplier of private capital, given by (26), is strictly negative if and only if $\sigma_Y < \underline{\sigma_Y}$*. Moreover, a necessary yet not sufficient condition is that* $\eta_K > \underline{\eta_K}$*.*

Proof. The necessity and sufficiency of $\sigma_Y < \underline{\sigma_Y}$ follows trivially from requiring that the expression in curly brackets of (26) be strictly negative. To show the necessity of $\eta_K > \underline{\eta_K}$, note that $\sigma_Y \geq 0$; $\sigma_Y < \underline{\sigma_Y}$ can only hold, therefore, if $\underline{\sigma_Y} > 0$. Solving this inequality for η_K delivers $\eta_K > \underline{\eta_K}$. □

Proposition 2 states that public investment crowds out private capital in the long run if private capital and labor are strong complements in production ('small' σ_Y) and if public capital augments private capital to a significant extent ('large' η_K). This case is depicted in Panel (c) of Figure 1 with solid lines. Conditional on the initial level of labor employment (i.e., $L = L_0$), public capital decreases the marginal productivity of private capital, so that the public investment increase shifts the $\dot{q} = 0|_0$ curve leftward. If $\theta_G < \overline{\theta_G}$, however, labor employment is higher in the long-run, which puts rightward pressure on the $\dot{q} = 0|_0$ curve. Under the conditions of Proposition 2, the leftward shift dominates, so that $\dot{q} = 0|_0$ moves to, say, $\dot{q} = 0|_1$, with a lower long-run stock of private capital. A labor-augmenting public capital technology or insufficiently productive public capital—even if capital-augmenting—both give rise to a positive private capital multiplier. In this case, the $\dot{q} = 0|_0$ moves to, say, $\dot{q} = 0|_H$, increasing private capital in the long run.

4 Transitional Effects of a Public Investment Shock

This section studies the transitional dynamics of a permanent public investment shock in the log-linearized model. I first describe the log-linearization procedure and the solution of the resulting log-linearized model in Section 4.1. Subsequently, Section 4.2 lays out the calibration strategy, and Section 4.3 discusses the results of the quantitative simulations.

4.1 Log-Linearization and Model Solution

I log-linearize the model around the initial steady state. Hereafter, I denote by x the initial steady-state value of a generic variable $x(t)$. I assume an initial steady state featuring zero net foreign assets (i.e., $F = 0$), which implies a balanced current account (i.e., $Z = 0$) and positive domestic assets (i.e., $A = qK > 0$). Most log-linearized variables are defined as $\tilde{x}(t) \equiv dx(t)/x$ and $\dot{\tilde{x}}(t) \equiv d\dot{x}(t)/x = \dot{x}(t)/x$. Exceptions include domestic and net foreign assets, whose log-linear versions are defined as $\tilde{A}(t) \equiv rdA(t)/Y$ and $\tilde{F}(t) \equiv rdF(t)/Y$; and lump-sum taxes, which are log-linearized as $\tilde{T}(t) \equiv dT(t)/Y$. The complete log-linearized model is presented in Appendix A.2.

I consider a *permanent* and *unanticipated* increase in public investment occurring at time $t = 0$—i.e., $\tilde{I}_G(t) = \tilde{I}_G$ for all $t \geq 0$. Public consumption is kept fixed at its initial level, so that $\tilde{C}_G(t) = 0$. Total public spending is financed by lump-sum taxes, implying that $\tilde{T}(t) = \omega_G^I \tilde{I}_G$, where $\omega_G^I \equiv I/Y$ denotes the public investment-to-GDP ratio. Given the public investment shock, public capital accumulates according to $\tilde{K}_G(t) = (1 - e^{-\sigma_G t})\tilde{I}_G$, where $\sigma_G \equiv I_G \Phi'_G(\cdot)/K_G > 0$ is the elasticity of the public capital installation function. The economy approaches the new steady state as $t \to \infty$. Variables at their new steady-state values are denoted by $x(\infty)$.

To solve the log-linearized model, I first combine equations (A.11), (A.12), and (A.15) (see Appendix A.2) in a static system for $\tilde{Y}(t)$, $\tilde{L}(t)$, and $\tilde{w}(t)$, conditional on the state variables $\tilde{K}(t)$, $\tilde{K}_G(t)$, and $\tilde{X}(t)$. The quasi-reduced form solution of this system can be written as

$$\begin{bmatrix} \tilde{Y}(t) \\ \tilde{L}(t) \\ \tilde{w}(t) \end{bmatrix} = \Upsilon \begin{bmatrix} \tilde{K}(t) \\ \tilde{X}(t) \\ \tilde{K}_G(t) \end{bmatrix}, \tag{27}$$

where Υ is a coefficient matrix defined as follows:

$$\Upsilon \equiv \frac{1}{\sigma_Y + \omega_{LL}\theta_K} \begin{bmatrix} \theta_K(\sigma_Y + \omega_{LL}) & -\theta_L\omega_{LL}\sigma_Y & \theta_G(\sigma_Y + \omega_{LL}) - (1 - \sigma_Y)\omega_{LL}\theta_L\eta_L \\ \theta_K\omega_{LL} & -\sigma_Y\omega_{LL} & \omega_{LL}[\theta_G - \eta_L(1 - \sigma_Y)] \\ \theta_K & \omega_{LL}\theta_K & \theta_G - \eta_L(1 - \sigma_Y) \end{bmatrix}$$

For future reference, denote the typical element of Υ by υ_{ij}, for $i = \{y, l, w\}$ and $j = \{k, x, g\}$.

The next step consists of setting up a dynamic system for the state variables of the model. Using equations (A.6)-(A.9) in conjunction with (A.13) and (A.16), this system can be written as

$$\begin{bmatrix} \dot{\tilde{K}}(t) \\ \dot{\tilde{q}}(t) \\ \dot{\tilde{X}}(t) \\ \dot{\tilde{A}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{r\omega_I}{\sigma_A\omega_A} & 0 & 0 \\ \frac{r\theta_K}{\sigma_Y\omega_A}(1-\upsilon_{yk}) & r & -\frac{r\theta_K}{\sigma_Y\omega_A}\upsilon_{yx} & 0 \\ 0 & 0 & r-\alpha & -\frac{r-\alpha}{\omega_A} \\ r\omega_w\upsilon_{wk} & 0 & r(\omega_w\upsilon_{wx}-\omega_X) & r \end{bmatrix} \begin{bmatrix} \tilde{K}(t) \\ \tilde{q}(t) \\ \tilde{X}(t) \\ \tilde{A}(t) \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma_q(t) \\ 0 \\ \gamma_A(t) \end{bmatrix}, \tag{28}$$

where $\omega_I \equiv I/Y$ is the ratio of private investment to GDP, $\sigma_A \equiv -(I/K)[\Phi''(\cdot)/\Phi'(\cdot)]$ is the elasticity of the marginal installation function of private capital, $\omega_A \equiv rA/Y$ denotes the ratio of income from financial assets to GDP, $\omega_w \equiv w/Y$ is the ratio of wages to GDP, and $\omega_X \equiv X/Y$ is the ratio of full consumption to GDP. The terms υ_{yk} and υ_{yx} correspond to the elements in the first row, first and second columns of Υ. Similarly, υ_{wk} and υ_{wx} correspond to the third row, first and second columns of Υ. Finally, $\gamma_q(t)$ and $\gamma_A(t)$ are the shock terms through which the public investment impulse affects the system:

$$\begin{aligned} \gamma_q(t) &\equiv \frac{r\theta_K}{\sigma_Y\omega_A}\left[\upsilon_{yg} - (1-\sigma_Y)\eta_K\right](1-e^{-\sigma_G t})\tilde{I}_G, \\ \gamma_A(t) &\equiv -r\left[\omega_w\upsilon_{wg}(1-e^{-\sigma_G t}) - \omega_G^I\right]\tilde{I}_G, \end{aligned}$$

where υ_{yg} and υ_{wg} correspond to the elements in the last column, first and third rows of Υ.

The stability properties of the system are governed by the eigenvalues of the 4×4 Jacobian matrix on the right hand-side of (28). The trace of this matrix is $3r - \alpha > 0$, while its determinant is strictly positive for $r > \alpha$ and $w > T$.[8] Hence, the Jacobian matrix possesses either four positive eigenvalues, or two negative and two positive eigenvalues (four negative eigenvalues are ruled out by the positive trace). Plausible calibrations of the model yield the latter case, with two negative and two positive real eigenvalues (see Section 4.2), implying a unique and locally saddle-path stable steady state.[9] Two special cases of (28)

[8] In particular, the determinant of the Jacobian matrix is $(r-\alpha)(\omega_w - \omega_T)\frac{r^3\omega_I\theta_K\theta_L}{\sigma_A\omega_A^3(\sigma_Y+\omega_{LL}\theta_K)} > 0$.

[9] Bom and Ligthart (2014a) show that distortionary taxation can lead to complex eigenvalues, in which case the dynamic responses to the public investment shock are cyclical. Under lump-sum taxes, and for

are worth noting. First, the system decouples into two recursive subsystems when labor supply is exogenous. In this case $\omega_{LL} = 0$, so that $\upsilon_{yx} = 0$; the capital market subsystem $\dot{\tilde{K}}(t)$–$\dot{\tilde{q}}(t)$ can then be solved independently of the consumption-saving subsystem $\dot{\tilde{X}}(t)$–$\dot{\tilde{A}}(t)$. Second, the system has a zero eigenvalue if $r = \alpha$ (in which case the third row of the Jacobian matrix only consists of zeros), which gives rise to a hysteretic steady state.

Solving the log-linearized model amounts to: (i) finding the impulse response functions of the state variables $\tilde{K}(t)$, $\tilde{q}(t)$, $\tilde{X}(t)$, and $\tilde{A}(t)$ in the dynamic system (28) to the public investment shock; (ii) using these impulse response functions to recover those of the static system (27); and (iii) obtaining the solution for the remaining variables (e.g., $\tilde{C}(t)$ and $\tilde{I}(t)$) using the relevant log-linearized expressions in Appendix A.2. The analytical derivation of the impulse response functions is provided in the technical appendix to this paper (see Bom, 2016). Here, I focus on the numerical simulations of the model.

4.2 Parameter Values

I calibrate the model for a typical small open economy in the euro area, assuming the economy is initially at its steady state with zero net foreign assets and net exports ($F = Z = 0$). Steady-state output is normalized to unity ($Y = 1$). I select values for the following parameters: β, δ, θ_G, σ_Y, $\bar{z}$, $\bar{z}_G$, and r; and for the following shares: $\omega_C \equiv C/Y$, $\omega_G^I \equiv I_G/Y$, $\omega_G^C \equiv C_G/Y$, K_G/Y, and $\omega_{LL} \equiv (1-L)/L$. As explained below, the remaining parameters follow in a recursive fashion. The relevant parameters values are summarized in Table 1.

The parameters/shares in the baseline calibration are obtained as follows. In line with eurozone averages, I set the private consumption, government consumption, and public investment ratios to GDP to $\omega_C = 0.56$, $\omega_G^C = 0.17$, and $\omega_G^I = 0.04$, respectively. From the government budget constraint (16), the tax revenues-to-GDP ratio is $\omega_T = \omega_G^C + \omega_G^I = 0.21$. Because net exports are initially zero, the private investment share of GDP is $\omega_I = 1 - \omega_C - \omega_G^I - \omega_G^C = 0.23$, which implies $I = \omega_I Y = 0.23$.

Following Bom and Ligthart (2014a), I specify the private capital installation function as $\Phi(x) \equiv \bar{z}\,[\ln(\bar{z} + x) - \ln(\bar{z})]$. The parameter $\bar{z}$ governs the degree of adjustment costs

plausible parameter values, this possibility does not arise.

to private capital and is set to $\bar{z} = 0.532$, which implies steady-state adjustments costs of 0.2% of GDP. The depreciation rate of private capital is set to $\delta = 10\%$, a common value in the literature. The initial stock of private capital follows from the accumulation function (4): $K = \left[(e^{\delta/\bar{z}} - 1)\bar{z}\right]^{-1} I = 2.091$. In turn, this implies, via (8), a steady-state Tobin's q value of $q = \bar{z}/(I/K + \bar{z}) = 1.207$. Assuming a world interest rate of interest of $r = 0.04$, the capital share of GDP in steady-state is thus $\theta_K = (rqK + I)/Y = 0.331$, which implies a labor share of $\theta_L = 1 - \theta_K = 0.669$. The elasticity of the private capital installation function is then $\sigma_A = (I/K)/(I/K + \bar{z}) = 0.171$.

I assume a benchmark leisure-labor ratio of $\omega_{LL} = 1$, which implies a Frisch elasticity of labor supply of the same magnitude. Although there is substantial controversy with respect to the empirical size of this parameter, unitary Frisch elasticities are commonly assumed (e.g., Drautzburg and Uhlig, 2015). The leisure-labor ratio determines steady-state labor employment as $L = 1/(1 + \omega_{LL}) = 0.5$; the wage rate then follows from the labor share as $w = \theta_L Y/L = 1.338$. To study the qualitative and quantitative importance of the elasticity of labor supply, I also consider the alternative values $\omega_{LL} = 0$ (exogenous labor supply) and $\omega_{LL} = 2$ (elastic labor supply).

Following the IMF (2014), the public capital-to-GDP ratio is set at 58%. Regarding the public capital installation function, I define $\Phi_G(x) \equiv \bar{z}_G \left[\ln(\bar{z}_G + x) - \ln(\bar{z}_G)\right]$ and assume $\bar{z}_G = \bar{z} = 0.532$. Given that $I_G/K_G = 0.069$, the depreciation rate of public capital follows from (17) in steady state: $\delta_G = \Phi_G(I_G/K_G) = 0.065$. Also, the elasticity of the public capital installation function is $\sigma_G = (\bar{z}_G I_G/K_G)/(\bar{z}_G + I_G/K_G) = 0.061$.

Concerning the technology parameters, I choose the baseline value of $\sigma_Y = 0.5$ for the elasticity of substitution between private factors, which corresponds to the middle value of Chirinko's (2008) interval of 0.4–0.6. For the output elasticity of public capital, I use Bom and Ligthart's (2014b) value of $\theta_G = 0.08$. In the baseline case where public capital is capital-augmenting, therefore, $\eta_K = \theta_G/\theta_K = 0.242$. As sensitivity analysis, I also consider the alternative values $\sigma_Y = 0.4$ and $\sigma_Y = 0.6$ for the elasticity of substitution, and $\theta_G = 0.05$ and $\theta_G = 0.12$ for the output elasticity of public capital. I also consider the alternative technology scenario of labor-augmenting public capital, in which case $\eta_L = \theta_G/\theta_L = 0.120$.

Turning to the household sector, the initial stock of financial assets is $A = qK = 2.523$, which implies steady-state full consumption of $X = rA + w - T = 1.229$. With respect to the mortality rate, I set $\beta = 0.018$, which reflects an average working life of 55 years. The pure rate of time preference is then determined by (19) as $\alpha = (rX - \beta^2 A)/(X + \beta A) = 0.038 < r$. The leisure share of full consumption is implied by the household's first-order condition (13): $1 - \varepsilon_C = w(1 - L)/X = 0.544$. The consumption share of full consumption is then $\varepsilon_C = 0.456$.

For the baseline parameter values, the threshold values defined in Proposition 1 are $\underline{\theta_G} = 0.020$ and $\overline{\theta_G} = 0.127$. Hence, the baseline output elasticity of public capital $\theta_G = 0.08$ falls in between these values, implying positive long-run effects of public investment on full consumption and labor. In terms of Proposition 2, one finds $\underline{\sigma_Y} = 0.633$ and $\underline{\eta_K} = 0.032$. The baseline values $\sigma_Y = 0.5 < \underline{\sigma_Y}$ and $\eta_K = 0.242 > \underline{\eta_K}$ thus imply long-run crowding out of private capital. Concerning the stability properties of the model, the eigenvalues of the Jacobian matrix of (28) are -0.016, -0.217, 0.257, and 0.058, so that the system is saddle-path stable.

4.3 Simulation Results

This section uses the parameterized model to simulate numerically the dynamic macroeconomic responses—in particular, the response of private capital formation—to a 10% increase in public investment. Section 4.3.1 discusses the results for the baseline parameterization. Subsequently, Section 4.3.2 conducts a sensitivity analysis to changes in key parameters. For intuition, I illustrate the dynamic effects of public investment in Figure 3.

4.3.1 Baseline Model

Figure 2 reports the impulse responses to a permanent public investment shock in the baseline parameterization (solid lines), varying only the labor supply elasticity to the alternative values of $\omega_{LL} = 2$ (dashed lines) and $\omega_{LL} = 0$ (dotted lines).

In the baseline model, Tobin's q strongly falls on impact, recovers in about eight years and raises above its initial value for approximately 20 years, falling again later and approaching its initial level from below. Relative to its initial level, therefore, private investment is crowded out during virtually the whole transitional process, showing only a very mild and short-lived increase about 12 years after the shock. Consequently, the stock of private capital exhibits a strongly non-monotonic transitional path, falling in the first eight years after the shock, partially recovering for about 20 periods, and then falling again gradually towards its new (lower) steady state level. Panel (a) of Figure 3 illustrates the transitional dynamics of Tobin's q and private capital. Starting at E, the economy jumps down to E_0 on impact and approaches E_∞ in the long run, after completing the dynamic path represented by the dotted arrow. That $\dot{q} = 0$ shifts to the left, crowding out private capital in the long run, follows straight from the conditions stated in Proposition 2, which hold for baseline parameter values (see Section 4.2). Understanding the impact effect and complex transitional dynamics towards the new steady state requires analyzing the response of labor employment, however.

To understand the short-run effect, first note that Tobin's q falls on impact even if labor supply were exogenous (see below), in view of the lower long-run marginal productivity of capital. In the endogenous case, however, a negative wealth effect on labor supply exacerbates the impact drop in Tobin's q and, therefore, the short-run crowding-out effect. The wealth effect arises as forward-looking households anticipate higher wages from gains in labor productivity. In Panel (b) of Figure 3, this wealth effect moves the economy from E to E_0 at the time of the shock. The impact increase in full consumption prompts households to increase goods consumption and cut on labor supply. In panel (c), the labor supply curve L^s shifts to $L^s(0)$, reducing labor and increasing wages. Because private capital is predetermined, the capital-labor ratio rises and private capital productivity falls, which lowers Tobin's q and reduces private capital formation.

Over time, as anticipated, the stock of public capital gradually builds up, increasing labor productivity and wages. In Panel (c) of Figure 3, the labor demand curve L^d starts a long, continuous shift to the right towards $L^d(\infty)$, inducing along the way both an intertemporal substitution effect and a wealth effect in labor supply. In a first transitional phase, lasting about 30 years, the intertemporal substitution effect dominates, so that labor employment expands and the capital-labor ratio falls, which stimulates Tobin's q and private capital formation. This phase comes to a halt about year 30 when, with high employment and wages, the wealth effect starts dominating. In Panel (c), the leftward shift of the labor supply curve dominates the rightward shift of the labor demand curve, which bends the dotted transitional path backwards. From this point onwards labor employment falls towards its new steady-state level, dragging private capital formation with it.

The importance of endogenous labor in the private capital response to public investment becomes apparent when one varies the labor supply elasticity. Increasing the labor supply elasticity to $\omega_{LL} = 2$, for instance, magnifies both the wealth effect and the intertemporal substitution effect. In Figure 2 (dashed lines), private investment is then more strongly crowded out on impact, but also recovers much faster, raising the private capital stock substantially above its initial level for a long transitional period. Exogenous labor supply (dotted lines), on the other hand, switches off both wealth and intertemporal labor supply effects, so that the crowding-out effect on private capital formation is weaker on impact, but also more persistent over time. In short, labor dynamics cause the non-monotonic private capital dynamics, which is exacerbated for larger labor supply elasticities.

4.3.2 Alternative Parameters

This section examines the sensitivity of the simulated impulse responses to changes in the values of key model parameters. Panel (a) of Figure 4 considers small deviations of the baseline elasticity of substitution $\sigma_Y = 0.5$ (solid lines) to the alternative values $\sigma_Y = 0.4$ (dashed lines) and $\sigma_Y = 0.6$ (dotted lines). Although both values are still below $\underline{\sigma_Y} = 0.633$—so that public investment is known beforehand to crowd out private capital in the long run—the transitional dynamics of private investment and private capital are qualitatively very different. Perhaps not surprisingly, the value $\sigma_Y = 0.4$ gives rise to stronger transitional crowding out. A small increase to $\sigma_Y = 0.6$, however, suffices to generate a long transitional crowding-in effect on private investment. Despite the crowding-out effect in the short and long run, private capital formation rises above its initial level for a long transitional phase in between. This result highlights the quantitative importance of the elasticity of substitution in shaping the effects of public investment on private capital information.

Panel (b) of Figure 4 studies how different assumptions about the factor-augmentation role of public capital affect the dynamics of private capital formation. In particular, it compares the baseline case of capital-augmentation (i.e., $\eta_K > \eta_L = 0$; solid lines) with the alternatives scenarios of factor-neutrality (i.e., $\eta_L = \eta_K$; dashed lines) and labor-augmentation (i.e., $\eta_L > \eta_K = 0$; dotted lines). The results are markedly different. Whereas private capital formation is crowded out during the entire transition between steady states in the baseline case, it is totally crowded in the cases of factor-neutrality and labor-augmentation. Moreover, these alternatives scenarios generate qualitatively similar results, differing mainly in the magnitude of the responses. Understanding the public-private investment relationship thus requires identifying the factor-augmentation role of public capital.

Finally, Panel (c) of Figure 4 varies the baseline output elasticity of public capital $\theta_G = 0.08$ (solid lines) to the alternative values of $\theta_G = 0.05$ (dashed lines) and $\theta_G = 0.12$ (dotted lines), which are both within the range of plausible values according to Bom and Ligthart (2014b). Note that both values fall within the interval ($\underline{\theta_G}$, $\overline{\theta_G}$)=(0.020, 0.127), so that the signs of the labor and full consumption multipliers are unaffected. Also, the corresponding values of η_K (0.151 and 0.363) both exceed $\underline{\eta_K} = 0.032$, so that the private capital multiplier remains negative. Despite the qualitatively similar long-run effects of public investment on private capital formation, the transitional dynamics are substantially different on both quantitative and qualitative grounds. In particular, although the higher elasticity $\theta_G = 0.12$ intensifies the crowding-out effect, a slightly smaller elasticity of $\theta_G = 0.05$ gives rise to moderate crowding-in effects in the first decades of transition (except in the very short run). Hence, the public-private investment relationship is also very sensitive to the magnitude of this parameter.

5 Concluding Remarks

This paper investigates the dynamic effects of public investment on private capital formation in a general equilibrium macroeconomic model of a small open economy. The model allows for factor-biased public capital by combining asymmetric factor-augmentation with a constant—but in general different than one—elasticity of substitution between private inputs. I show that a permanent impulse to public investment crowds out public capital in the long run when public capital directly augments private capital and the elasticity of substitution is smaller than one. This cases arises for plausible calibrations of the model.

This paper also shows that the dynamics of private capital formation to a public investment shock are rather complex, falling in the short and in the long run, but potentially rising during transition between steady states. The crucial element for this transitional non-monotonicity lies in the dynamics of labor employment, which interplays with private capital via factor complementarity in production. The dynamic effects of public investment on private capital formation are therefore very sensitive to several key parameters for which little empirical consensus exists, namely the elasticity of substitution between private capital and labor, the elasticity of labor supply, and the output elasticity of public capital. Small variations within plausible ranges of values give rise to rather different, both quantitatively and qualitatively, dynamics of private capital formation. The complex dynamics of private capital formation and its sensitivity on key parameters may help explain the mixed empirical evidence on the private-public investment relationship.

Table 1: Baseline Parameter Values

Description	Parameter/Share	Value
Selected values		
Private consumption-to-GDP ratio	$\omega_C \equiv C/Y$	0.560
Government consumption-to-GDP ratio	$\omega_G^C \equiv C_G/Y$	0.170
Public investment-to-GDP ratio	$\omega_G^I \equiv I_G/Y$	0.040
Parameter of the private capital installation function	$\bar{z}$	0.532
Depreciation rate of private capital	δ	0.100
Rate of interest	r	0.040
Leisure-labor ratio	$\omega_{LL} \equiv (1-L)/L$	1.000
Public capital-to-GDP ratio	K_G/Y	0.580
Output elasticity of public capital	θ_G	0.080
Parameter of the public capital installation function	$\bar{z}_G$	0.532
Elasticity of substitution capital/labor	σ_Y	0.500
Birth/death rate	β	0.018
Implied values		
Tax revenues-to-GDP ratio	$\omega_T \equiv T/Y$	0.210
Private investment-to-GDP ratio	$\omega_I \equiv I/Y$	0.230
Output elasticity of labor	θ_L	0.669
Elasticity of the private installation function	σ_A	0.171
Depreciation rate of public capital	δ_G	0.065
Elasticity of public installation function	σ_G	0.061
Capital-augmentation elasticity	η_K	0.242
Pure rate of time preference	α	0.038
Preference weight of consumption in utility	ε_C	0.456

Figure 1: Long-Run Effects of a Public Investment Impulse

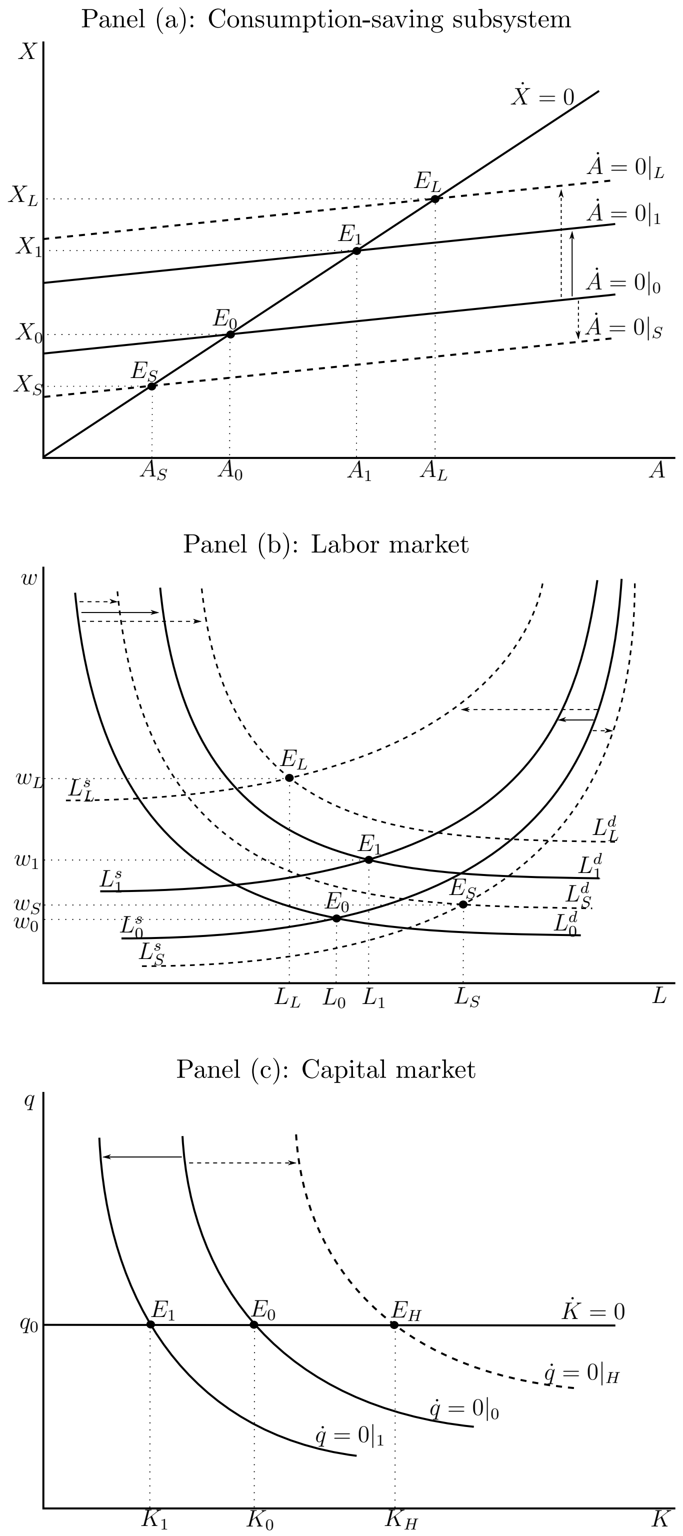
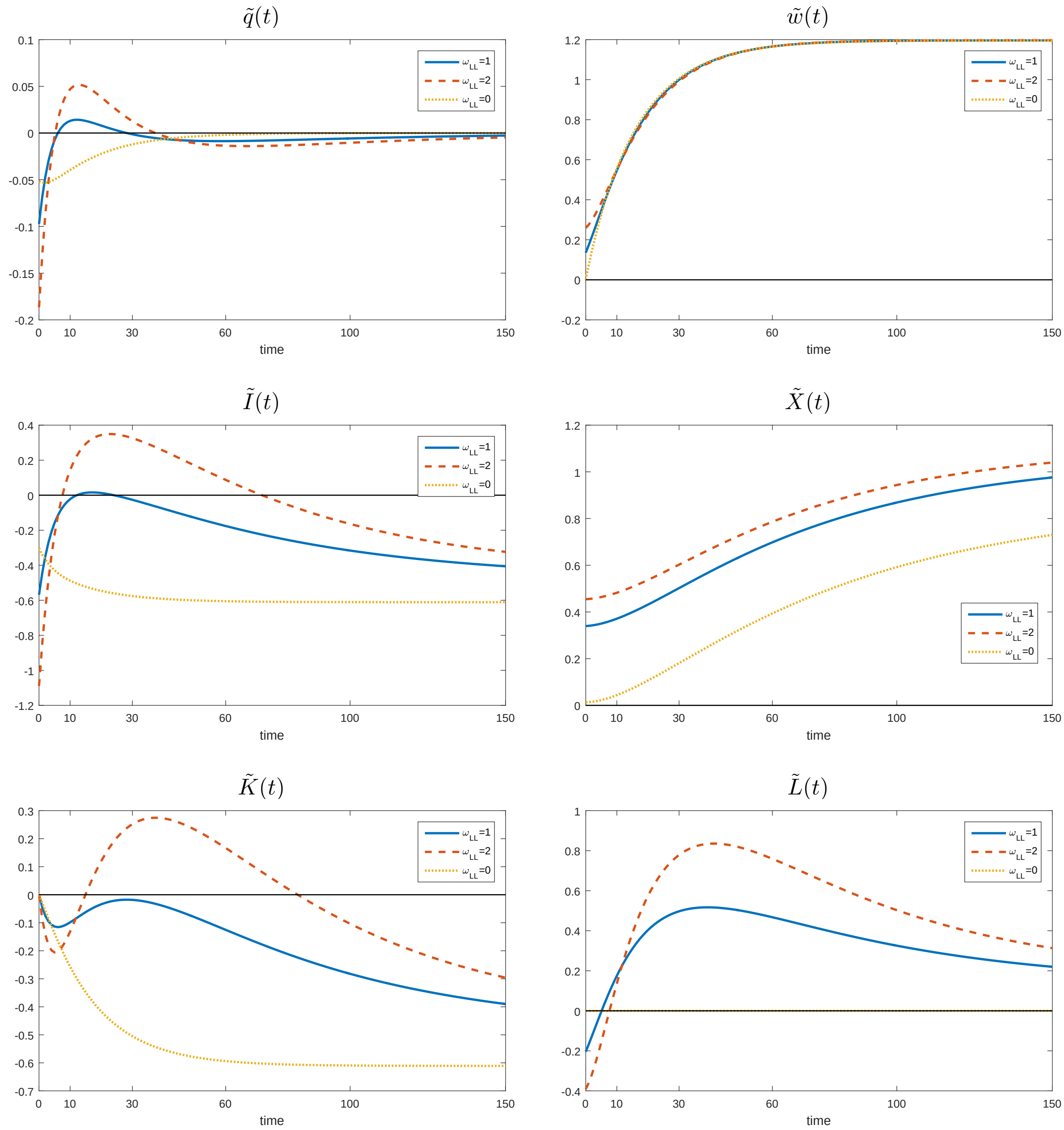


Figure 2: Dynamic Effects of a Public Investment Shock: Baseline Model



Notes: The time unit is a year. For each variable, the vertical axis measures the percentage deviation from the the steady-state value. The public investment shock amounts to $\tilde{I}_G = 0.1$. The baseline parameter values are reported in Table 1.

Figure 3: Transitional Effects of a Capital-Augmenting Public Investment Impulse

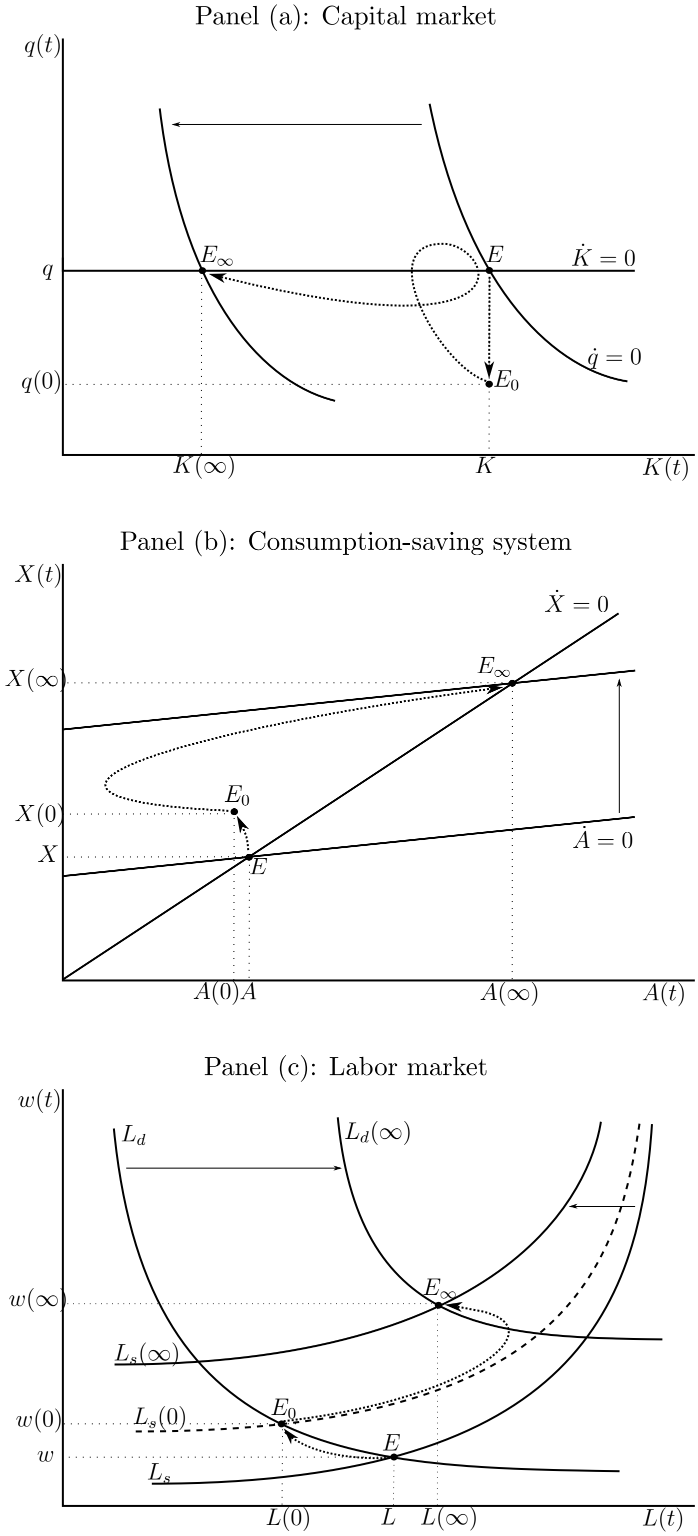
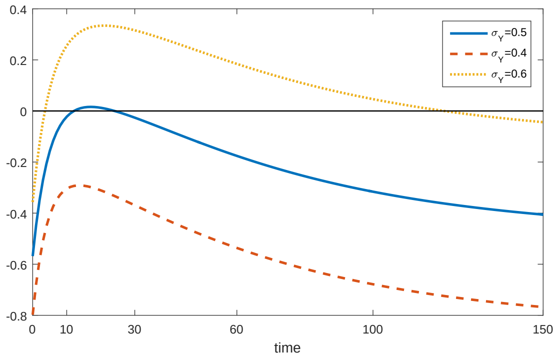


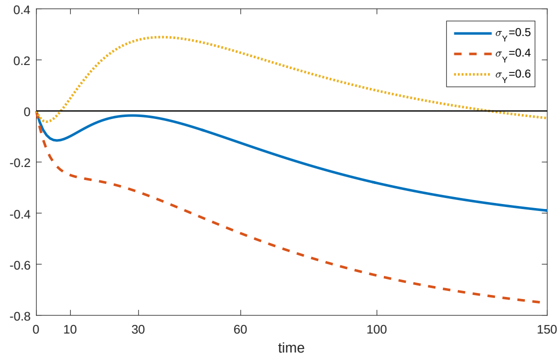
Figure 4: Dynamic Effects of a Public Investment Shock: Alternative Parameter Values

Panel (a): Alternative values of σ_Y

$\tilde{I}(t)$

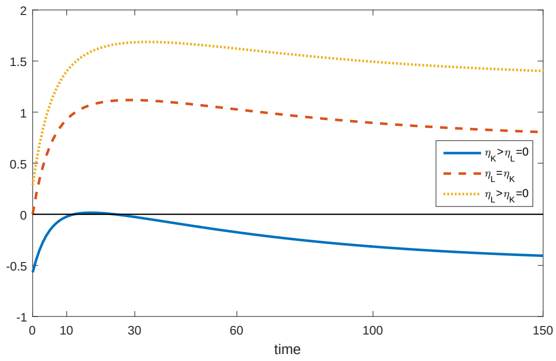


$\tilde{K}(t)$

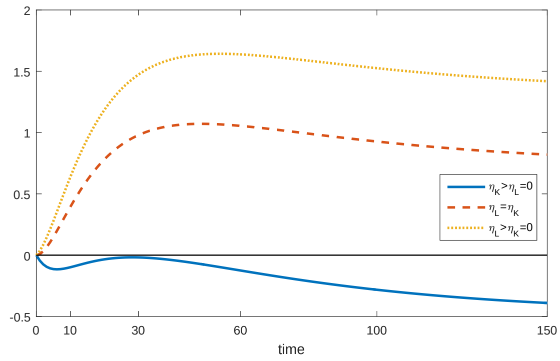


Panel (b): Different types of factor-augmentation

$\tilde{I}(t)$

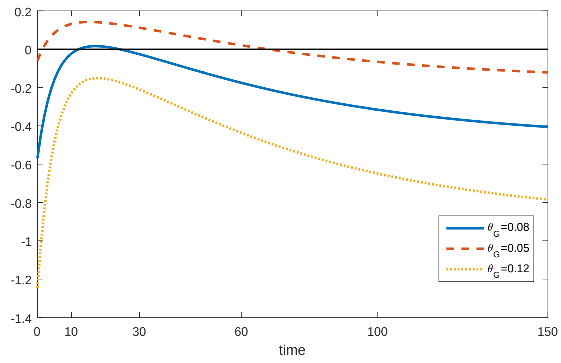


$\tilde{K}(t)$

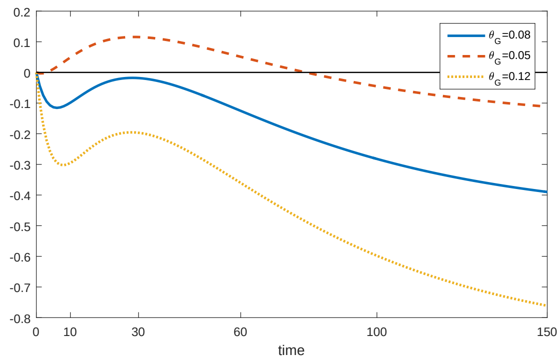


Panel (c): Alternative values of θ_G

$\tilde{I}(t)$



$\tilde{K}(t)$



Notes: The time unit is a year. For each variable, the vertical axis measures the percentage deviation from the the steady-state value.

Appendix

A.1 Reduced-Form Long-Run Multipliers

This section derives the reduced-form long-run multipliers of public investment. Fully differentiating (7) with respect to public investment gives an expression for the long-run change in wages:

$$\frac{dw}{dI_G} = \frac{Y_L\theta_K}{\sigma_Y K}\frac{dK}{dI_G} + \frac{Y_L[\theta_G - \eta_L(1-\sigma_Y)]}{\sigma_Y K_G}\frac{dK_G}{dI_G} - \frac{Y_L\theta_K}{\sigma_Y L}\frac{dL}{dI_G}. \tag{A.1}$$

Solving the steady-state conditions $\dot{A} = 0$ and $\dot{X} = 0$ gives steady-state full consumption as linear function of $w - T$. Fully differentiating this function with respect to I_G gives the quasi-reduced form long-run multiplier of full consumption:

$$\frac{dX}{dI_G} = \frac{\omega_X}{(1+\omega_{LL})\theta_L - \omega_T}\left(\frac{dw}{dI_G} - 1\right). \tag{A.2}$$

Denote by $L = L(K, K_G, X)$ the equilibrium level of labor implicitly defined by the steady-state labor demand and labor supply functions, L^d and L^s. A quasi-reduced form expression for the long-run multiplier of labor obtains by fully differentiating $L(K, K_G, X)$ with respect to I_G:

$$\frac{dL}{dI_G} = \frac{\omega_{LL}}{\omega_{LL}\theta_K + \sigma_Y}\left\{\theta_K\frac{L_0}{K_0}\frac{dK}{dI_G} + [\theta_G - \eta_L(1-\sigma_Y)]\frac{L_0}{K_{G0}}\frac{dK_G}{dI_G} - \sigma_Y\frac{L_0}{X_0}\frac{dX}{dI_G}\right\}. \tag{A.3}$$

Because q and I/K are fixed in the long run, so is the marginal productivity of private capital. By totally differentiating Y_K with respect to K, K_G, and L, setting the resulting expression to zero, and then solving for dK/dI_G, one arrives at

$$\frac{dK}{dI_G} = \frac{K_0}{\theta_L K_{G0}}[\theta_G - \eta_K(1-\sigma_Y)]\frac{dK_G}{dI_G} + \frac{K_0}{L_0}\frac{dL}{dI_G}. \tag{A.4}$$

Noting that the long-run change in the capital stock is a multiple of the public investment change—i.e., $dK_G/dI_G = K_{G0}/I_{G0} = 1/\Phi^{-1}(\delta_G)$— equations (A.1)–(A.4) form a linear system of four equations in four unknowns: dw/dI_G, dK/dI_G, dL/dI_G, and dX/dI_G.

The solution of this system for dw/dI_G is

$$\frac{dw}{dI_G} = \frac{\theta_G(1+\omega_{LL})}{\omega_G^I}. \tag{A.5}$$

The solutions for dX/dI_G, dL/dI_G, and dK/dI_G correspond to (20), (23), and (26) in the main text.

A.2 Log-Linearized Model

The model is is log-linearized around a steady state with $F = 0$, which implies $A = qK$. For most variables, I define $\tilde{x}(t) \equiv dx(t)/x$ and $\dot{\tilde{x}}(t) \equiv d\dot{x}(t)/x$. The exceptions are $A(t)$ and $F(t)$, which are defined as $\tilde{z}(t) \equiv rdz(t)/Y$ and $\dot{\tilde{z}}(t) \equiv rd\dot{z}(t)/Y$, for $z = \{A, F\}$.

Using those notational conventions, the dynamic equations (4), (6), (10), (15), and (17) are log-linearized as follows:

$$\begin{aligned}
\dot{\tilde{q}}(t) &= r\tilde{q}(t) - \frac{r\theta_K}{\sigma_Y\omega_A}[\tilde{Y}(t) - \tilde{K}(t) - (1-\sigma_Y)\eta_K\tilde{K}_G(t)], && \text{(A.6)}\\
\dot{\tilde{K}}(t) &= \frac{r\omega_I}{\omega_A}[\tilde{I}(t) - \tilde{K}(t)], && \text{(A.7)}\\
\dot{\tilde{A}}(t) &= r\left[\tilde{A}(t) + \omega_w\tilde{w}(t) - \tilde{T}(t) - \omega_X\tilde{X}(t)\right], && \text{(A.8)}\\
\dot{\tilde{X}}(t) &= (r-\alpha)\left[\tilde{X}(t) - \frac{\tilde{A}(t)}{\omega_A}\right], && \text{(A.9)}\\
\dot{\tilde{K}}_G(t) &= \sigma_G[\tilde{I}_G - \tilde{K}_G(t)], && \text{(A.10)}
\end{aligned}$$

where $\omega_A \equiv rA/Y$, $\omega_I \equiv I/Y$, $\omega_w \equiv w/Y$, $\omega_X \equiv X/Y$, and $\sigma_G \equiv I_G\Phi_G'(\cdot)/K_G$.

In turn, the log-linearized expressions of static equations (1), (7), (8), (12), (13), (16) and $A(t) = q(t)K(t) + F(t)$ are:

$$\tilde{Y}(t) = \theta_K \tilde{K}(t) + \theta_L \tilde{L}(t) + \theta_G \tilde{K}_G(t), \tag{A.11}$$

$$\tilde{w}(t) = \frac{1}{\sigma_Y}\left[\tilde{Y}(t) - \tilde{L}(t) - (1-\sigma_Y)\eta_L \tilde{K}_G(t)\right], \tag{A.12}$$

$$\tilde{q}(t) = \sigma_A[\tilde{I}(t) - \tilde{K}(t)], \tag{A.13}$$

$$\tilde{C}(t) = \tilde{X}(t), \tag{A.14}$$

$$\tilde{L}(t) = \omega_{LL}[\tilde{w}(t) - \tilde{X}(t)], \tag{A.15}$$

$$\tilde{T}(t) = \omega_G^I \tilde{I}_G + \omega_G^C \tilde{C}_G, \tag{A.16}$$

$$\tilde{F}(t) = \tilde{A}(t) - \omega_A[\tilde{q}(t) + \tilde{K}(t)], \tag{A.17}$$

where $\sigma_A \equiv -(I/K)(\Phi''(\cdot)/\Phi'(\cdot))$, $\omega_{LL} \equiv (1-L)/L$, $\omega_G^I \equiv I_G/Y$, and $\omega_G^C \equiv C_G/Y$.

References

ABIAD, A., D. FURCERI, AND P. TOPALOVA (2015): "The Macroeconomic Effects of Public Investment: Evidence from Advanced Economies," IMF Working Paper, International Monetary Fund, WP/15/95.

AFONSO, A. AND M. ST. AUBYN (2009): "Macroeconomic Rates of Return of Public and Private Investment: Crowding-In and Crowding-Out Effects," *The Manchester School*, 77, 21–39.

BARCZYK, D. (2016): "Ricardian Equivalence Revisited: Deficits, Gifts and Bequests," *Journal of Economic Dynamics and Control*, 63, 1–24.

BAXTER, M. AND R. G. KING (1993): "Fiscal Policy in General Equilibrium," *American Economic Review*, 83, 315–334.

BLANCHARD, O. J. (1985): "Debt, Deficits, and Finite Horizons," *Journal of Political Economy*, 93, 223–247.

BOM, P. R. AND J. E. LIGTHART (2014a): "Public Infrastructure Investment, Output Dynamics, and Balanced Budget Fiscal Rules," *Journal of Economic Dynamics and Control*, 40, 334–354.

——— (2014b): "What Have We Learned from Three Decades of Research on the Productivity of Public Capital?" *Journal of Economic Surveys*, 28, 889–916.

BOM, P. R. D. (2016): "Factor-Biased Public Capital and Private Capital Crowding Out: Technical Appendix," mimeo, Deusto Business School.

BOUAKEZ, H., M. GUILLARD, AND J. ROULLEAU-PASDELOUP (2014): "Public Investment, Time to Buid, and the Zero Lower Bound," Série des Documents de Travail No. 2014-03, CREST.

CAVALLO, E. AND C. DAUDE (2011): "Public Investment in Developing Countries: A Blessing or a Curse? ," *Journal of Comparative Economics*, 39, 65–81.

Chirinko, R. S. (2008): "σ: The long and Short of It," *Journal of Macroeconomics*, 30, 671–686.

Clancy, D., P. Jacquinot, and M. Lozej (2016): "Government Expenditure Composition and Fiscal Policy Spillovers in Small Open Economies Within a Monetary Union," 48, 305–326.

Cooper, R. W. and J. C. Haltiwanger (2006): "On the Nature of Capital Adjustment Costs," *Review of Economic Studies*, 73, 611–633.

Dioikitopoulos, E. V. and S. Kalyvitis (2008): "Public Capital Maintenance and Congestion: Long-Run Growth and Fiscal Policies," *Journal of Economic Dynamics and Control*, 32, 3760–3779.

Drautzburg, T. and H. Uhlig (2015): "Fiscal Stimulus and Distortionary Taxation," *Review of Economic Dynamics*, 18, 894–920.

Eden, M. and A. Kraay (2014): "'Crowding in' and the Returns to Government Investment in Low-Income Countries," Policy Research Working Paper No. 6781, World Bank.

Fisher, W. H. and S. J. Turnovsky (1998): "Public Investment, Congestion, and Private Capital Accumulation," *Economic Journal*, 108, 399–413.

Heijdra, B. J. and L. Meijdam (2002): "Public Investment and Intergenerational Distribution," *Journal of Economic Dynamics and Control*, 26, 707–735.

IMF (2014): *World Economic Outlook: Legacies, Clouds, Uncertainties*, Washington: International Monetary Fund.

Leduc, S. and D. Wilson (2012): "Roads to Prosperity or Bridges to Nowhere? Theory and Evidence on the Impact of Public Infrastructure Investment," *NBER Macroeconomics Annual*, 27, 89–142.

Leeper, E. M., T. B. Walker, and S.-C. S. Yang (2010): "Government Invesment and Fiscal Stimulus," *Journal of Monetary Economics*, 57, 1000–1012.

Perotti, R. (2004): "Public Investment: Another (Different) Look," mimeo, Università Bocconi.

Schmitt-Grohé, S. and M. Uribe (2003): "Closing Small Open Economy Models," *Journal of International Economics*, 61, 163–185.

Turnovsky, S. J. and W. H. Fisher (1995): "The Composition of Government Expenditure and Its Consequences for Macroeconomic Performance," *Journal of Economic Dynamics and Control*, 19, 747–786.

Voss, G. M. (2002): "Public and Private Investment in the United States and Canada," *Economic Modelling*, 19, 641–664.