Optimal Growth Models

Steve Ambler
Université Mohammed VI Polytechnique

Fall 2023

Introduction

- Objectives of this part of the course:
  1. Study two basic optimal growth models: Ramsey-Cass-Koopmans (RCK) and overlapping generations (OG)
  2. Stress methodological aspects with we will be able to apply to more complex models: transforming variables to induce stationarity, welfare theorems, etc.
  3. For the Ramsey-Cass-Koopmans model I follow Romer (2018) but in **discrete time**
  4. I find it easier to distill economic intuition from optimality conditions in discrete time with Lagrange multipliers, instead of continuous-time Hamiltonians
  5. I'm a big fan of the economic interpretation of Lagrangian multipliers

RCK: households

- Representative household's utility:

$$U = E_t \sum_{i=0}^{\infty} \beta^i \frac{{C_{t+i}}^{(1-\theta)}}{1-\theta} \frac{L_{t+i}}{H},$$

with C the consumption of each household member, L total population, and H the number of households

- Flow budget constraint at time t:

$$w_t \frac{L_t}{H} + R_t \frac{K_t^s}{H} = C_t \frac{L_t}{H} + \frac{I_t}{H}.$$

, with K the aggregate capital stock, I aggregate investment expenditure, w the real wage and R the capital rental rate

Laws of motion

- Capital stock:

$$K^s_{t+1} = (1-\delta)K^s_t + I_t.$$

- Population:

$$L_{t+1} = (1+n)L_t, \quad n > 0.$$

Aggregate production function

- Of the general form

$$Y_t = s_t \left(A_t L_t\right)^{\alpha} K_t^{(1-\alpha)}$$

- We can model technical progress by a **deterministic** trend or a **stochastic** trend
- Deterministic trend with productivity shocks:

$$A_t = (1+g)A_{t-1}, \quad g > 0,$$

$$s_t = {s_{t-1}}^{\rho} \exp(v_t), \quad 0 < \rho < 1,$$

with v_t an unpredictable (white noise) shock
- Stochastic trend:

$$\ln(A_t) = g + \ln(A_{t-1}) + v_t,$$

$$s_t = 1,$$

Solow residuals

- From the production function, taking logs,

$$\ln(Y_t) - \alpha \ln(L_t) - (1-\alpha)\ln(K_t) = \ln(s_t) - \alpha \ln(A_t) \equiv Z_t$$

- With a deterministic trend, take logs and replace $\ln(A_{t-1})$ we have

$$\begin{aligned} Z_t &= \ln(s_t) + \alpha(\ln(1+g) + \ln(A_{t-1})) \\ &\approx \ln(s_t) + \alpha(g + \ln(A_{t-1})) \\ &= \ln(s_t) + \alpha(g + g + \ln(A_{t-2})) \\ &= \ldots = \ln(s_t) + \alpha(t \times g + \ln(A_0)) \end{aligned}$$

- In this case, log productivity is stationary around a linear time trend
- After a shock, log productivity reverts to the linear trend given by $\alpha(t \times g + \ln(A_0))$

Solow residuals 2

- With a stochastic trend we have

$$Z_t = \alpha \left(g + \ln(A_{t-1}) + v_t \right)$$

- In this second case, log productivity has a **unit root**. A v_t shock leads log productivity to deviate **permanently** from the line given by $A_0 + g \cdot t$
- This is a **fundamental** distinction in time series analysis. When hit by a shock, variables with a unit root (I(1) in the parlance of econometricians) are **permanently** affected

Representative firm

- Representative competitive firm maximizes profits given by

$$\pi_t = Y_t - w_t L_t^d - R_t K_t^d,$$

with L_t^d is its labour demand and K_t^d its demand for capital

- The individual (or household) holds the capital stock and undertakes investment, and the firm just **rents** capital from households. This makes its problem **static**

Representative firm 2

- The maximization problem gives the following consequences of the first order conditions (FOCs):

$$w_t = \alpha s_t {A_t}^{\alpha} {L_t^d}^{(\alpha-1)} {K_t^d}^{(1-\alpha)},$$

$$R_t = (1-\alpha) s_t \left(A_t L_t^d \right)^{\alpha} {K_t^d}^{-\alpha}.$$

- The real wage equals the marginal product of labour
- the marginal product of capital equals the **rental rate** of capital R_t

Household's problem

- Writing out the infinite summation we get the Lagrangian

$$\max_{C_{t+i}\,,\,K^s_{t+1+i}\,,\,\lambda_{t+i}\,,\,\forall i\geq 0} \mathcal{L} = E_t \Bigg\{ \frac{{C_t}^{(1-\theta)}}{1-\theta} \frac{L_t}{H}$$

$$+\lambda_t \left(w_t \frac{L_t}{H} + R_t \frac{K^s_t}{H} + (1-\delta) \frac{K^s_t}{H} - C_t \frac{L_t}{H} - \frac{K^s_{t+1}}{H} \right)$$

$$+\beta \frac{{C_{t+1}}^{(1-\theta)}}{1-\theta} \frac{L_{t+1}}{H}$$

$$+\beta\lambda_{t+1} \left(w_{t+1} \frac{L_{t+1}}{H} + R_{t+1} \frac{K^s_{t+1}}{H} + (1-\delta) \frac{K^s_{t+1}}{H} - C_{t+1} \frac{L_{t+1}}{H} - \frac{K^s_{t+2}}{H} \right)$$

$$+ \ldots \Bigg\}.$$

Household's problem 2

- Note that K_t^s is **not** a choice variable at time t. It is **predetermined**
- The household has a **dynamic** problem
- The FOC for consumption is:

$$E_t \left[\beta^i {C_{t+i}}^{-\theta} \frac{L_{t+i}}{H} - \beta^i \lambda_{t+i} \frac{L_{t+i}}{H} \right] = 0, \quad \forall i$$

- This gives:

$$E_t {C_{t+i}}^{-\theta} = E_t \lambda_{t+i}$$

- The FOC for K_{t+1} gives:

$$E_t \left\{ -\lambda_{t+i} + \beta \left(\lambda_{t+i+1} \left[(1-\delta) + R_{t+i+1} \right] \right) \right\} = 0$$

- As is usually the case, the FOCs for λ_{t+i} gives the sequence of budget constraints

Household's problem 3

- The last equation gives the standard **Euler equation** for consumption once we substitute C_{t+i} for λ_{t+i} using the FOC for C_{t+i}. If we look at time t, we can drop expectations for variables dated t to get

$$C_t{}^{-\theta} = E_t\left(\beta\left[(1-\delta)+R_{t+1}\right]C_{t+1}{}^{-\theta}\right)$$

- If there is no uncertainty we get immediately

$$\Rightarrow C_{t+1} = \left(\beta\left[(1-\delta)+R_{t+1}\right]\right)^{1/\theta}C_t$$

- This gives us the **slope** of the time path of consumption
- If β is high (close to one) and the individual is relatively **patient**, or if the rental rate of capital R_{t+1} is high, the consumption path with slope **up**, with $C_{t+1} > C_t$
- Vice versa (obviously)

Equilibrium conditions

$$L_t = L_t^d,$$

$$K_t^s = K_t^d = K_t,$$

$$Y_t + (1-\delta)K_t = C_t L_t + K_{t+1}$$

- The firm's FOCs give

$$w_t = \alpha Y_t / L_t,$$

$$R_t = (1-\alpha) Y_t / K_t$$

- These two conditions immediately give:

$$w_t L_t + R_t K_t = \alpha Y_t + (1-\alpha) Y_t = Y_t,$$

- This is just an example of **Euler's Theorem**. The payments to factors exhaust the economy's output

Euler's theorem

- Here's a simple proof of Euler's theorem for general homogeneous production functins. Let's say we have (for arbitrary $\gamma > 0$)

$$\gamma Y = F(\gamma K, \gamma L)$$

- Differentiate w.r.t. γ to get

$$\frac{\partial(\gamma Y)}{\partial \gamma} = Y = \frac{\partial F}{\partial(\gamma K)} K + \frac{\partial F}{\partial(\gamma K)} K$$

- Set $\gamma = 1$ and we have the Euler's theorem we usually refer to in economics: payments to factors exhaust output

Sources of non-stationarity and variable transformation

- Sources of non-stationarity in the model:
  1. Population growth ($n > 0$);
  2. Productivity growth ($g > 0$).
- Define

$$c_t \equiv C_t / A_t,$$

(consumption per unit of effective labour, and

$$k_t \equiv \frac{K_t}{A_t L_t}$$

- To facilitate a graphical analysis of the dynamics of the economy, we have to transform variables to make them stationary
- Note that if we use Lagrangian techniques (discrete time), we can do this **after** calculating optimality conditions
- If we use **dynamic programming** (more on this in the course on the RBC model — if we have time), we have to use "Bellman equations" to use "stationary discounted dynamic programming," which means transforming variables **before** deriving optimality conditions. In order to do so, we have to know **a priori** which transformed variables are stationary
- We use perfect foresight to simulate (drop expectations) and suppose deterministic technical progress with $s_t = 1$

Tranformed equation system

- Substitute Cs in the Euler equation, drop expectations, and use the FOC for K_{t+1} to substitute out R_{t+1}:

$$\frac{C_t{}^{-\theta}}{C_{t+1}{}^{-\theta}} = \beta\left[(1-\delta) + (1-\alpha)\left(A_{t+1}L_{t+1}\right)^{\alpha} K_{t+1}{}^{-\alpha}\right]$$

- Divide the numerator and denominator on the left hand side by $A_t{}^{-\theta}$, and use $A_t = A_{t+1}/(1+g)$ to get

$$\frac{(C_t/A_t)^{-\theta}}{(C_{t+1}/(A_{t+1}/(1+g)))^{-\theta}} = \beta\left[(1-\delta) + (1-\alpha)k_{t+1}{}^{-\alpha}\right]$$

- Take logs (and use $\ln(1+g) \approx g$) to get

$$-\theta \ln(c_t) + \theta \ln(c_{t+1}) + \theta g = \ln\beta + \ln\left((1-\delta) + (1-\alpha)k_{t+1}{}^{-\alpha}\right)$$

$$\Rightarrow \ln(c_{t+1}) - \ln(c_t) = \frac{\ln\beta + \ln\left((1-\delta) + (1-\alpha)k_{t+1}{}^{-\alpha}\right) - \theta g}{\theta}$$

- This is a nonlinear law of motion that says that the growth of consumption depends on the level of the capital stock

Transformed equation system 2

- Now we need a simple law of motion for k_t
- From the law of motion for capital and the resource constraint:

$$Y_t = C_t L_t + K_{t+1} - (1-\delta)K_t$$

$$\Rightarrow \frac{Y_t}{A_t L_t} = \frac{C_t}{A_t} + \frac{K_{t+1}}{A_t L_t} - (1-\delta)\frac{K_t}{A_t L_t}$$

- From the aggregate production function we have $\frac{Y_t}{A_t L_t} \equiv y_t = {k_t}^{1-\alpha}$, so we get

$$
\begin{aligned}
{k_t}^{(1-\alpha)} &= c_t + \frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_t L_t} - (1-\delta)k_t \\
&= c_t + \frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_t L_t} - (1-\delta)k_t \\
&= c_t + k_{t+1}\frac{(1+g)A_t + (1+n)L_t}{A_t L_t} - (1-\delta)k_t
\end{aligned}
$$

Transformed equation system 3

- Continuing,

$$k_t{}^{(1-\alpha)} = c_t + k_{t+1}(1+g)(1+n) - (1-\delta)k_t$$

$$\Rightarrow k_t{}^{(1-\alpha)} \approx c_t + (1+n+g)k_{t+1} - (1-\delta)k_t,$$

where we use $ng \approx 0$ if n and g are small

$$\Rightarrow (1+n+g)k_{t+1} - (1-\delta)k_t = k_t{}^{(1-\alpha)} - c_t$$

$$\Rightarrow k_{t+1} - k_t = k_t{}^{(1-\alpha)} - \delta k_t - (n+g)k_{t+1} - c_t$$

- This is a nonlinear law of motion which says that the growth rate of the capital stock depends on the capital stock and on consumption

Transformed equation system 4

- So we have the following two equations:

$$\ln c_{t+1} - \ln c_t = \frac{\ln\left((1-\delta) + (1-\alpha)k_{t+1}^{-\alpha}\right) + \ln\beta - \theta g}{\theta},$$

$$k_{t+1} - k_t = k_t^{(1-\alpha)} - \delta k_t - (n+g)k_{t+1} - c_t$$

- The first equation is equivalent to (2.25) dans Romer, and the second is equivalent to (2.26), except that they are both in discrete time
- The graphical analysis in Romer is in continuous time. How do we go from discreate to continuous time?
- Think of continuous time as the limit of a process where we shrink the length of the time periods (annual to quarterly to monthly to weeekly to dailly to . . .)

Transformed equation system 5

- Figure 2.1 in Romer shows the dynamics of c_t as a function of the levels of c_t and k_t. Figure 2.2 shows the dynamics of k_t as a function of the levels of c_t and k_t
- Putting the two together gives Figure 2.3. Figure 2.4 shows that there are **convergent** and **divergent** dynamic paths
- Choosing the convergent path of the system amounts to choosing an appropriate **transversality condition** on the household's problem
- We will need to analyze the curves along which $\dot{c}_t = 0$ and $\dot{k}t = 0$, to which we turn

Phase diagram

- The two fundamental equations are

$$k_{t+1} - k_t = k_t^{(1-\alpha)} - \delta k_t - (n+g)k_{t+1} - c_t$$

$$\ln c_{t+1} - \ln c_t = \frac{\ln\left((1-\delta) + (1-\alpha)k_{t+1}^{-\alpha}\right) + \ln\beta - \theta g}{\theta}$$

- In the steady state we get

$$0 = k^{*(1-\alpha)} - \delta k^* - (n+g)k^* - c^*$$

$$0 = \frac{\ln\left((1-\delta) + (1-\alpha)k^{*-\alpha}\right) + \ln\beta - \theta g}{\theta}$$

Phase diagram2

- The first equation is a nonlinear relationship between k^* and c^* along which k_t is constant, like the one in Figure 2.2
- The second an equation gives a unique value for k^*, a vertical line along which c_t is constant, like the one in Figure 2.1
- We can figure out what happens on either side of the two curves by staring at the equations. This gives the arrows in Figure 2.3 in the four quadrants delineated by the four curves
- Figure 2.4 shows that for most (almost all) initial conditions the economy **diverges**
- There is a unique **convergent** path towards the steady state

Phase diagram3

- It may help to think of a horse saddle, from which we get the term "saddlepoint stability"
- If we let a marble go at exactly the right spot on the saddle (on the convergent path), it will roll towards the middle of the saddle
- Let it go anywhere else and it will fall off the saddle to the left or to the right
- What allows us to argue that, given an initial situation, the economy will start from a point along the **convergent arm** of the saddle path?
- Note that at a point in time the capital stock k_t is **predetermined**
- On the other hand, c_t is not. Consumption must satisfy the **Euler equation** and also a **transversality condition** which says essentially that the value of the "costate" variable (c_t here) does not explode

Balanced growth path

- The capital stock per unit of effective labour converges to k^*
- When $k = k^*$, $K = kAL$, so the capital stock grows at the rate $(n + g)$
- With CRS, output also grows at the rate $(n + g)$
- Output per worker Y/L and capital per worker K/L grow at rate g
- On the balanced growth path, the growth of output per worker depends only on the rate of technical progress g

- For any variable x_t we define

$$\hat{x}_t \equiv \frac{x_t - \bar{x}}{\bar{x}}$$

where $\bar{x}$ is the deterministic steady state level
- We are measuring variables as proportional deviations from steady state
- For the capital stock equation, we have

$$(1+n)(1+g)k_{t+1} = (1-\delta)k_t + s_t k_t^{(1-\alpha)} - c_t$$

$$\Rightarrow (1+n)(1+g)(k_{t+1} - \bar{k}) = (1-\delta)(k_t - \bar{k}) + (1-\alpha)\bar{s}\bar{k}^{-\alpha}(k_t - \bar{k})$$
$$+\bar{k}^{1-\alpha}(s_t - \bar{s}) - (c_t - \bar{c}) + O.T.$$
$$\Rightarrow (1+n)(1+g)\hat{k}_{t+1} = (1-\delta)\hat{k}_t$$
$$+(1-\alpha)\bar{s}\bar{k}^{-\alpha}\hat{k}_t + \bar{k}^{-\alpha}\bar{s}\hat{s}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t + O.T.$$

Linearization and stochastic analysis 2

- Similarly, we can linearize the consumption equation
- After some algebra, we get

$$\hat{c}_t = E_t \hat{c}_{t+1} + a_1 \hat{k}_{t+1} + a_2 \hat{s}_t + O.T.$$

$$\hat{k}_{t+1} = b_1 \hat{k}_t + b_2 \hat{s}_t + b_3 \hat{c}_t + O.T.$$

$$\hat{s}_t = \rho \hat{s}_{t-1} + v_t + O.T.$$

- In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ -b_2 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{s}_t \\ \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & b_1 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{s}_{t-1} \\ \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_t \tag{1}$$

- This is of the form analyzed in Ambler (2001), which shows in detail how to simulate this type of model (using Matlab, Octave, R, etc.)

Linearization and stochastic analysis 4

- It's also possible to take the nonlinear equation system, assign numerical values to the parameters, and use *Dynare* to simulate the model. *Dynare* will calculate the steady state, linearize the equations, and simulate for given probability distributions of the shocks
- *Dynare* can also calculate **higer order approximations** of the model and simulate those. This can make a difference for **welfare analysis** since, with non-linearities, the **stochastic means** of variables can differ from their **deterministic** steady state values
- With a few small changes we have the system from Farmer (1999, Chapter 3), which emphasizes the distinction between **regular equilibria** and **irregular equilibria**, where the number of stable roots of the system exceeds the number of predetermined variables. This leads to the possibility of multiple equilibria and "sunspot equilibria"
- Uhlig (1999) gives shortcuts to linearize the equations of this type of system

Rational expectations and cross-equation restrictions

- Parameters of the basic model: α, β, δ, θ, ρ, g, n
- Parameters of the linear system of equations: a_1, a_2, b_1, b_2, b_3, ρ
- Parameters we can estimate with auxiliary equations: g, n
- So the 6 parameters of the linearized system of equations are a function of only 5 fundamental parameters of the model. Rational expectations impose nonlinear restrictions on the values of the 6 parameters of the linearized system

Welfare theorems

- Suppose there is a social planner who maximizes the welfare of the representative household, subject to the global resource constraint:

$$\max_{C_{t+i}\,,\,K^s_{t+1+i}\,,\,\lambda_{t+i}\,,\,\forall i\geq 0} \mathcal{L} = E_t\left\{ \frac{{C_t}^{(1-\theta)}}{1-\theta}\frac{L_t}{H}\right.$$

$$+\lambda_t\left(s_t(A_tL_t)^{\alpha}{K_t}^{(1-\alpha)} + (1-\delta)\frac{K_t}{H} - C_t\frac{L_t}{H} - \frac{K_{t+1}}{H}\right)$$

$$+\beta\frac{{C_{t+1}}^{(1-\theta)}}{1-\theta}\frac{L_{t+1}}{H}$$

$$+\beta\lambda_{t+1}\left(s_{t+1}(A_{t+1}L_{t+1})^{\alpha}{K_{t+1}}^{(1-\alpha)} + (1-\delta)\frac{K_{t+1}}{H} - C_{t+1}\frac{L_{t+1}}{H}\right.$$

$$\left.\left. - \frac{K_{t+2}}{H}\right) + \ldots\right\}.$$

Welfare theorems 2

- The FOCs for C_{t+i} and K_{t+i+1} are

$$C_{t+i} : E_t \left[\beta^i C_{t+i}{}^{-\theta} \frac{L_{t+i}}{H} - \beta^i \lambda_{t+i} \frac{L_{t+i}}{H} \right] = 0, \quad \forall i,$$

$$K_{t+i+1} :$$

$$E_t \left\{ -\lambda_{t+i} + \beta \left(\lambda_{t+i+1} \left[(1-\delta) + s_{t+i+1} \left(K_{t+i+1} / (A_{t+i+1} L_{t+i+1}) \right)^{-\alpha} \right] \right) \right\}$$

- Once we impose equilibrium conditions on the optimality conditions of agents when we explicitly calculate the general equilibrium, we get just these two equations
- So, the FCOs of the planner's problem give **exactly** the same system of equations as the preceding analysis
- This shows that, under certain conditions, we can find the competitive equilibrium as the solution to a suitably defined planning problem

Welfare theorems 3

- Conditions for the equivalence:
  1. Perfect competition;
  2. No externalities;
  3. Absence of distortions (for example distortionary taxes);
  4. With uncertainty, the existence of complete financial markets à la Arrow-Debreu.
  5. Everybody knows **everything**: structure of the economy and its parameters, statistical laws governing the shocks¨

Overlapping generations (Diamond model)

- I follow the presentation in Romer (Chapter 2 Part B) with a slight change in notation
- We'll be using a subjective discount rate β instead of a rate of time preference $1/(1+\rho)$

Overlapping generations 2: assumptions

1. $L_{t+1} = (1+n)L_t$ as before. In this section, it's better to interpret this as maximization by the individual and not the household
2. $A_{t+1} = (1+g)A_t$ as before. We'll only be looking at deterministic growth and not technology shocks
3. Same assumptions concerning the aggregate production function as in the model with an infinite-horizon representative agent
4. Individuals' planning horizons are 2 periods. The individual works in the first period. (See Farmer (1999, Section 6.8) for a technique to reduce models with longer horizons to 2-period models.)
5. We will assume a depreciation rate (δ) of 100%. This is relatively realistic in a model with time periods of roughly 30 years

Overlapping generations 3: preferences

- Utility function:

$$U_t = \frac{{C_{1t}}^{(1-\theta)}}{1-\theta} + \beta \frac{{C_{2t+1}}^{(1-\theta)}}{1-\theta},$$

  with $\theta > 0$, $0 < \beta < 1$

- Consumption in period 2 given by

$$C_{2t+1} = (1 + r_{t+1})(w_t A_t - C_{1t}),$$

  where w_t is the real wage **per unit of effective labour**

- Intertemporal budget constraint given by

$$C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = A_t w_t.$$

Overlapping generations 4: Maximization problem

- The individual maximizes

$$\mathcal{L} = \frac{{C_{1t}}^{(1-\theta)}}{1-\theta} + \beta \frac{{C_{2t+1}}^{(1-\theta)}}{1-\theta} + \lambda \left[A_t w_t - C_{1t} - \frac{1}{1+r_{t+1}} C_{2t+1} \right]$$

- The multiplier λ of course gives the value (in units of marginal utility) of an additional unit of output (in present value terms)
- The FOCs are

$${C_{1t}}^{-\theta} - \lambda = 0,$$

$$\beta {C_{2t+1}}^{-\theta} - \frac{1}{1+r_{t+1}} \lambda = 0$$

$$\Rightarrow \beta {C_{2t+1}}^{-\theta} = \frac{1}{1+r_{t+1}} {C_{1t}}^{-\theta}.$$

- Consequences of FOCs

$$C_{2t+1} = \left(\beta(1 + r_{t+1})\right)^{1/\theta} C_{1t}$$

- This is just a standard Euler equation for consumption
- Substituting in the budget constraint gives

$$C_{1t} + \beta^{1/\theta} \left(1 + r_{t+1}\right)^{(1-\theta)/\theta} C_{1t} = A_t w_t$$

$$\Rightarrow C_{1t} = \frac{1}{1 + \beta^{1/\theta} \left(1 + r_{t+1}\right)^{(1-\theta)/\theta}} A_t w_t.$$

- Consumption when young is a function (complicated in terms of paramters) of **lifetime income** given by $A_t w_t$

Overlapping generations 6

- Define $s(r)$ as the fraction of income saved (the **marginal propensity to save**). We have

$$s(r_{t+1}) \equiv \frac{A_t w_t - C_t}{A_t w_t}$$

$$= \frac{\beta^{1/\theta} (1 + r_{t+1})^{(1-\theta)/\theta}}{1 + \beta^{1/\theta} (1 + r_{t+1})^{(1-\theta)/\theta}},$$

$$C_{1t} \equiv [1 - s(r_{t+1})] A_t w_t.$$

- We get

$$\frac{\partial (1 + r_{t+1})^{(1-\theta)/\theta}}{\partial r_{t+1}} = \frac{(1-\theta)}{\theta} (1 + r_{t+1})^{(1-2\theta)/\theta}$$

- So, s is increasing in r if $\theta > 1$ and decreasing in r if $\theta < 1$. If $\theta = 1$ (for example, with log preferences), s is independent of r

Law of motion for k_t

- We have

$$K_{t+1} = s(r_{t+1}) L_t A_t w_t$$

- Divide each side by $L_{t+1} A_{t+1}$ to get

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t$$

- We can write

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s\left(f'(k_{t+1})\right)\left[f(k_t) - k_t f'(k_t)\right]$$

- With log utility ($\theta = 1$) and Cobb-Douglas production we have

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{\beta}{1+\beta} (1-\alpha) {k_t}^{\alpha} \equiv D {k_t}^{\alpha}$$

Law of motion for k_t 2

- This is the equivalent of (2.61) in Romer
- Here, α is the share of capital in national revenue. We get Figure 2.10 in Romer. In the long run,

$$k^* = D{k^*}^{\alpha} \Rightarrow k^* = D^{1/(1-\alpha)}.$$

- Taking a first-order approximation of the law of motion for k_t we get

$$k_{t+1} \approx k^* + \alpha D{k^*}^{(\alpha-1)}(k_t - k^*)$$

$$\Rightarrow k_{t+1} - k^* = \alpha D \left(D^{(1/1-\alpha)}\right)^{(\alpha-1)} = \alpha(k_t - k^*)$$

- With $\alpha = 1/3$, k eliminates $2/3$ of the gap w.r.t. its long-run value each period

Dynamics in the general case

- We have

$$k_{t+1} = \left[\frac{1}{(1+n)(1+g)}\right]\left[s\left(f'(k_{t+1})\right)\right]\left[\frac{[f(k_t) - k_t f'(k_t)]}{f(k_t)}\right]\left[f(k_t)\right]$$

- From right to left the four terms represent:
  1. production per unit of effective labour in t;
  2. the share of production paid to capital;
  3. the share of capital income which is saved; and
  4. the ratio of effective labour in t to effective labour in $t+1$
- Figure 2.12 in Romer gives different possibilities for the functional relationship between k_t and k_{t+1}. There is the possibility of multiple equilibria

Dynamic inefficiency

- With log utility, Cobb-Douglas production, and $g = 0$, we have (along a balanced growth path)

$$k^* = \frac{1}{(1+n)} \frac{\beta}{1+\beta} (1-\alpha) {k_t}^{\alpha}$$

$$\Rightarrow k^* = \left[\frac{1}{(1+n)} \frac{\beta}{1+\beta} (1-\alpha) \right]^{1/(1-\alpha)}$$

- Therefore, the marginal product of capital on the balanced growth path is

$$f'(k^*) = \alpha k^{*(\alpha-1)} = \frac{\alpha}{1-\alpha} (1+n) \frac{1+\beta}{\beta}.$$

- The golden-rule capital stock is given by

$$f'(k_{GR}) = (1+n).$$

Dynamic inefficiency 2

- This is the capital stock which maximises consumption per worker in the steady state. We maximize

$$f(k^*) - (1+n)k^*,$$

with nk^* the level of investment necessary to keep capital per worker constant

- Note that the book gives

$$f'(k_{GR}) = n$$

There may be a mistake: it would be compatible depreciation of 0%

- In general,

$$C_{t+1} = f(k_t)A_tL_t - (1-\delta)K_{t+1}$$

$$\Rightarrow c_{t+1} = f(k_t) - k_{t+1}(1+n)(1+g) + (1-\delta)k_t.$$

Dynamic ineffeciency 3

- In the steady state,

$$c^* \approx f(k^*) - (n + g + \delta)k^*.$$

- With $g = 0$ and $\delta = 1$,

$$c^* = f(k^*) - (1 + n)k^*.$$

- In principle, k^* can be greater or less than k_{GR}. If $k^* > k_{GR}$, we can increase everyone's consumption (Figure 2.14 in Romer)
- The infinite number of generations gives a **social planner** a way of giving older persons more consumption that markets are incapable of doing
- Individuals, **must** save to consume in old age, even if the rate of return on capital is low

Government in the Diamond model

- If public spending is financed by lump sum taxation, with log utility and Cobb-Douglas production we have

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{\beta}{1+\beta} \left[(1-\alpha) {k_t}^{\alpha} - G_t \right].$$

- G_t just takes resources and reduces net output
- We can use Figure 2.14 to analyze the impact of a permanent change in G_t
- Government spending acts like a negative **additive** technology shock

Financing of public spending

- We have

$$k_{t+1} + b_{t+1} = \frac{1}{(1+n)(1+g)} \frac{\beta}{1+\beta} \left[(1-\alpha) {k_t}^{\alpha} - T_t \right].$$

- By reducing taxes, consumption increases and capital accumulation goes down

This version: **02/10/2023**