Answers to Romer Exercises (Part 3)

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Exercise 2.9

- The question revolves around an analytical solution to the Ramsey model when $\alpha = \theta$
- This is (of course!) more complicated than I thought when I assigned this problem
- I would not expect you to solve this in the context of an exam
- The solution involves a trick, which is a transformation of variables which leads to a differential equation which is autonomous (the rate of change of a variable depends only on the variable itself and parameters)

Exercise 2.9 (cont.)

- From (2.25) and (2.26) in Romer, using the Cobb-Douglas production function, we have
- We have

$$\frac{\dot{c}}{c} = \frac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta}$$

and

$$\dot{k} = y - c - (n+g)k \Rightarrow \frac{\dot{k}}{k} = \frac{y}{k} - \frac{c}{k} - (n+g)$$

Exercise 2.9: (a), (b)

- Setting $\dot{c} = 0$ and using bars for long-term values, we get

$$\alpha \bar{k}^{\alpha-1} = \rho + \theta g$$

$$\Rightarrow \bar{k} = \left(\frac{\rho + \theta g}{\alpha} \right)^{1/(\alpha-1)}$$

- Setting $\dot{k} = 0$ we get

$$\bar{c} = \bar{y} - (n+g) = \bar{k}^{\alpha} - (n+g) = \left(\frac{\rho + \theta g}{\alpha} \right)^{\alpha/(\alpha-1)} - (n+g)$$

- Define $z \equiv k/y$ and $x \equiv c/k$
- Using dots to signify time derivatives, We have

$$\dot{z} = \left(\frac{\dot{k}}{y}\right) = \frac{\dot{k}}{y} - \frac{k}{y^2}\dot{y}$$

and

$$\dot{x} = \left(\frac{\dot{c}}{k}\right) = \frac{\dot{c}}{k} - \frac{c}{k^2}\dot{k}$$

$$\Rightarrow \frac{\dot{x}}{x} = \frac{\dot{c}}{kx} - \frac{c}{xk^2}\dot{k}$$

$$= \frac{\dot{c}}{kc/k} - \frac{c}{k^2 c/k}\dot{k}$$

$$= \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$$

(c) (cont.)

- Substituting in the expressions for $\frac{\dot{c}}{c}$ and $\frac{\dot{k}}{k}$ and using the production function we get

$$\Rightarrow \frac{\dot{x}}{x} = \frac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta} - \frac{k^{\alpha}}{k} + \frac{c}{k} + (n+g)$$

$$= \frac{\alpha k^{\alpha-1} - \rho - \theta g}{\theta} - k^{\alpha-1} + x + (n+g)$$

$$= \left(\frac{\alpha}{\theta} - 1\right) k^{\alpha-1} - \frac{\rho}{\theta} + x + (n+g)$$

- Setting $\alpha = \theta$ we get

$$\Rightarrow \frac{\dot{x}}{x} = -\left(\frac{\rho - \theta g}{\theta} + (n+g)\right) + x$$

- This is an autonomous differential equation for x. It is unstable, since $\dot{x}$ depends positively on x

(c) (cont.)

- For z we have (using the production function)

$$\dot{z} = \frac{\dot{k}}{k}\frac{k}{y} - \frac{k}{y}\frac{\dot{(k^{\alpha})}}{k^{\alpha}}$$

$$= \frac{\dot{k}}{k}z - z\frac{\alpha k^{\alpha-1}\dot{k}}{k^{\alpha}} = z\frac{\dot{k}}{k} - \alpha z\frac{\dot{k}}{k} = (1-\alpha)z\frac{\dot{k}}{k}$$

$$= (1-\alpha)z\left(\frac{y-c}{k} - (n+g)\right) = (1-\alpha)z\left(\frac{1}{z} - x - (n+g)\right)$$

$$= (1-\alpha) - ((1-\alpha)x + (n+g))\,z$$

(c) (cont.)

- We have now expressed $\dot{z}$ and $\frac{\dot{x}}{x}$ in terms of z, x, and parameters of the model, as asked
- For a given x (we are told in part (d) to assume that x is constant along the saddle path), this is a (**stable**) nonhomogeneous linear differential equation

(d)

- The differential equation for z is of the form

$$z'(t) + az(t) = b$$

with $z(t_0) = z_0$ and with $a = ((1-\alpha)x + (n+g))$ and $b = (1-\alpha)$

- It has the solution

$$z(t) = (z_0 - b/a)\, e^{a(t_0 - t)} + b/a$$

- See `https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/fle/t`
- This explains my general preference for working in discrete time instead of continuous time. There are methods to guess and verify the solution to differential equations like this one (see Osborne's notes) but I don't find them intuitive.

(d) (cont.)

- Applying this solution in the current context gives the following solution for z:

$$z = \left(z_0 - \frac{(1-\alpha)}{((1-\alpha)x + (n+g))} \right) e^{((1-\alpha)x+(n+g))(t_0-t)} + \frac{(1-\alpha)}{(1-\alpha)x}$$

$$\Rightarrow z = \left(z_0 - \frac{(1-\alpha)}{((1-\alpha)x + (n+g))} \right) e^{((1-\alpha)x+(n+g))(t_0-t)} + \frac{1}{x}$$

(d) (cont.)

- The equation

$$\frac{\dot{x}}{x} = -\frac{\rho}{\theta} + x$$

is an example of a "logistic equation." It has the general solution (see Smith (2006) page 6, cited in Romer: see reference below)

$$x = \frac{\bar{x}}{1 + \bar{x}Ae^{\bar{x}t}}$$

- If $A > 0$ then x would fall to zero over time, driving marginal utility and k to infinity. If $A < 0$ then x would become negative in finite time. If $A = 0$ the x is constant and the consumer's transversality condition is satisfied. So the conjecture that x is constant has to hold

(e)

- So z shrinks exponentially towards its long run equilibrium at a constant rate. But we have

$$\frac{\dot{z}}{z} = \frac{\dot{k}}{k} - \frac{\dot{y}}{y}$$

$$\Rightarrow \frac{\dot{y}}{y} = \frac{\dot{k}}{k} - \frac{\dot{z}}{z}$$

- Since the model is just a special case of (2.25) and (2.26), the laws of motion are satisfied.

Exercise 2.10

- The key to answering this question is the idea that the tax on investment income affects the rate of return on capital.
- The household's Euler equation governs the time profile of consumption. This depends on the rate of return on capital
- As Romer notes in Section 2.6 (2019) edition:

  "Since the evolution of k is determined by technology rather than preferences, ρ [and therefore $(1-\tau)f'(k)$] enters the equation for $\dot{c}$ but not that for k. Thus only the $\dot{c} = 0$ locus is affected."

- See equations (2.25) and (2.26). ρ and $f'(k)$ enter equation (2.25) with opposite signs. From the point of view of the dynamics, a tax on investment acts exactly like a drop in the marginal productivity of capital

Exercise 2.10 (cont.)

- The impact of a tax on investment has very similar (but opposite) effects to the effects of a fall in the discount rate analyzed in Section 2.6
- So reading through Section 2.6 (and changing the sign of the shifts) goes a long way to answering this question, at least Parts (a) through (d)

Exercise 2.10 (e) (ii)

- The answer to this part of the question depends on whether or not foreigners have to pay a withholding tax on their foreign investment income
- In any case, equation (2.25) would imply that in long-run equilibrium (so that $\dot{c} = 0$, and ignoring growth so that $g = 0$), we must have

$$(1 - \tau_i) f'(k_i) = \rho$$

for country i

- If foreigners must also pay the tax on investment income, the rate of return on capital net of the tax will be equalized across countries in the long run, provided all consumers have the same rate of time preference ρ

Exercise 2.10 (f)

- We've seen previously (see Section 2.7) that what counts is not the time path of taxation but the present value of government spending
- So now the investment income tax will be combined with an increase in G means we must merge the analysis of a change in the discount rate (Section 2.6) with the analysis of an increase in government spending (Section 2.7)

Exercise 2.18 (a)

- There is no production
- There is no need to accumulate capital when young to rent out to firms when old
- Individuals when young have only one decision: how much of A to consume and how much to store to maximize utility
- So there will be no trade across generations

Exercise 2.18 (a) (cont.)

- The amount stored by the individual is $(A - C_{1t})$ This yields $x(A - C_{1t})$ units which can be consumed by individuals who are old in period 2
- So the budget constraint is

$$A = C_{1t} + \frac{1}{x} C_{2t+1}$$

- Note, as indicated in Footnote 26, x acts like the real rate of interest in equation (2.46) in Romer
- The individual's maximization problem is

$$\max_{\lambda, C_{1t}, C_{2t+1}} U = \ln(C_{1t}) + ln(C_{2t+1}) + \lambda \left(A - C_{1t} + \frac{1}{x} C_{2t+1} \right)$$

Exercise 2.18 (a) (cont.)

- The first order conditions are

$$\frac{1}{C_{1t}} - \lambda = 0,$$

$$\frac{1}{C_{2t+1}} - \frac{\lambda}{x} = 0$$

  and the first order condition with respect to λ gives the constraint
- Taking the ratio of these two conditions gives the Euler equation

$$C_{2t+1} = xC_{1t}$$

Exercise 2.18 (a) (cont.)

- In equilibrium, each young individual consumes $\frac{1}{2}A$ when young and $\frac{x}{2}A$ when old
- Substituting in the budget constraint gives

$$C_{1t} = \frac{1}{2}A$$

- As in the Ramsey-Cass-Koopmans model, the savings rate is independent of the rate of return to storage (with log utility)

Exercise 2.18 (b)

- In period t there are $L_0 \times (1+n)^t$ young individuals alive and $L_0 \times (1+n)^{(t-1)}$ old individuals alive
- So total consumption is given by

$$\frac{1}{2}AL_0(1+n)^t + \frac{x}{2}AL_0(1+n)^{(t-1)}$$

$$= \frac{1+n+x}{2}AL_0(1+n)^{(t-1)}$$

Exercise 2.18 (b) (cont.)

- A social planner would want to maximize lifetime consumption per capita.
- If storage is costly ($0 < x < 1$), then $\frac{1+x}{2}A$ is less than A.
- We could increase per capita lifetime consumption by forcing people to store nothing, and distributing current output to young and old individuals
- Of course this only works if the economy continues forever (infinite number of agents). If the economy ends, there would be no young individuals in the last period with endowments to redistribute to the old
- Because marginal utility is decreasing in consumption, the planner would want to equalize consumption by the old and young in each period

Exercise 2.18 (b) (cont.)

- Total production in each period would equal total production, which is

$$AL_0(1+n)^t$$

- This is clearly greater than total consumption, which is given above:

$$AL_0(1+n)^t > \frac{1+n+x}{2} AL_0(1+n)^{(t-1)}$$

$$\Leftrightarrow 1+n > \frac{1+n+x}{2} \Leftrightarrow 2+2n > 1+n+x,$$

which is clearly the case if $x < 1$

Exercise 2.18 (b) (cont.)

- Consumption per individual will be total consumption divided by population alive:

$$\frac{AL_0(1+n)^t}{L_0(1+n)^t + L_0(1+n)^{(t-1)}}$$

$$= \frac{A}{1+(1+n)^{-1}} = \frac{1+n}{2+n}A > \frac{1}{2}A$$

so consumption increases for both young and old individuals compared to the market solution

Reference

Smith, William (2006), "A Closed Form Solution to the Ramsey Model." *B.E. Journal in Macroeconomics (Contributions)* 6, Article 3

- I've posted a copy of this on my website for those who are interested

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