Notes on the Forward-Backward Simulation Method*

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Abstract

We show how to simulate dynamic, stochastic general equilibrium (DSGE) models numerically using the techniques developed by Blanchard and Kahn (1980) as applied to real business cycle models by King, Plosser and Rebelo (1987, 1988a, 1988b). We do this in order to make some of the steps of the algebra more explicit than in these papers, and to make easier the connection between the algebra and the computer code required to implement it. We also discuss how to simulate models that have not been reduced to their minimal state-space representations. We also consider how to extend the technique to models that are written down without eliminating extraneous state variables.

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1 Introduction

We consider in this short paper how to simulate numerically a model using the techniques developed by Blanchard and Kahn (1980) as applied to real business cycle models by King, Plosser and Rebelo (1987, 1988a, 1988b). We do this in order to make some of the steps of the algebra more explicit than in these papers, and to make easier the connection between the algebra and the computer code required to implement it.

In Blanchard and Kahn, the models studied were arbitrary, and the "non-predetermined state variables" were often price variables such as exchange rates. In King, Plosser and Rebelo, for versions in which the competitive equilibrium is calculated as the solution to a social planning problem, the non-predetermined state variables are most often the Lagrange multipliers of the planning problem, and can be thought of as "costate" variables. The technique can also be applied to models in which the general equilibrium is calculated explicitly, by solving for the first order conditions of individual agents, aggregating, and imposing conditions for market clearing.

In Blanchard and Kahn, the state space form of the model often comes from a model which is specified in an *ad hoc* fashion. In King, Plosser and Rebelo, the state space form comes from linearizing the first order conditions of a suitably defined optimization problem. See the technical appendix (1987) to their paper for details on the linearization procedure, which is essentially just a first-order Taylor expansion around the steady-state values of the model's variables, with the variables themselves measured in logs so that changes are *proportional* deviations from the steady state. See also the paper by Uhlig (1999) for shortcuts which simplify the work of linearizing. Canned algorithms such as Dynare (Juillard, 2005) can solve for the steady state numerically and calculate a linear or higher-order approximation to the model around that steady state numerically.

It is also possible to incorporate trending or I(1) variables. Such variables are assumed to be predetermined. They will be associated with unit roots. There are several

consequences of this. First, I(1) variables do not have unique steady-state values. Temporary shocks will have permanent effects on them. Rather than linearizing around a unique steady state, it is necessary to choose arbitrary initial values for I(1) variables. One must be very careful when doing so. If the DSGE model has a balanced growth path, a natural approach would be to use values for I(1) variables that are on the steady-state growth path: if there are variables that have initial values that are far away from the steady-state growth path, the solution to the model, including for stationary values, may wander far from the point around which it is linearized and the linear approximation may break down. Second, the unconditional moments of such variables do not exist, although conditonal moments at any finite horizon do. Third, stochastic simulations of systems with unit roots can have variables (especially the I(1) variables themselves) that wander arbitrarily far away from their initial conditions.

2 Steady State or Balanced Growth Path

For one-sector models it is often possible to solve analytically for the steady state. For multi-sectoral or multi-country models, it is often necessary to resort to algorithms for the numerical solution of nonlinear equation systems.

In all cases, especially when heavy use is made of numerical techniques to solve for the steady state, it is desirable to check the accuracy of the solution. The saddlepoint stability properties of the Blanchard and Kahn algorithm and indeed all algorithms that linearize around a steady state are typically only valid locally.

3 Linearization

[To be added.]

3.1 Application to the Neoclassical Growth Model

Let's use the version of the neoclassical growth model presented in King and Rebelo (2000). The aggregate production function is

$$Y_t = A_t F\left(K_t\ ,\ N_t X_t\right), \tag{1}$$

where A_t is a stationary productivity shock and X_t measures trend labor-augmenting technical progress, with

$$X_t = \gamma X_{t-1}, \qquad \gamma > 1. \tag{2}$$

The economy's overall resource constraint is

$$Y_t = C_t + I_t. \tag{3}$$

Capital accumulation is given by

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{4}$$

Preferences are given by an intertemporal utility function that can be written

$$U_t = E_t \left(\sum_{i=0}^{\infty} b^i U\left(C_t\ ,\ L_t\right) \right), \qquad b > 0, \tag{5}$$

where C_t is aggregate consumption, L_t is leisure, and E_t is the conditional expectations operator. We impose the normalization

$$N_t + L_t = 1, \tag{6}$$

where N_t is aggregate hours worked. King and Rebelo discuss the restrictions on the utility function in order for a balanced growth path to exist with aggregate hours

remaining stationary, and giving a transformed utility function

$$U_t = E_t \left(\sum_{i=0}^{\infty} \beta^i u\left(c_t\ ,\ L_t\right) \right), \qquad \beta > 0, \tag{7}$$

with $c_t \equiv C_t/X_t$. Dividing the production function equation through by X_t (the production function is assumed to be linearly homogeneous) gives the following system of equations in addition to the utility function:

$$y_t = c_t + i_t; \tag{8}$$

$$N_t = 1 - L_t; \tag{9}$$

$$\gamma k_{t+1} = (1-\delta)\, k_t + i_t; \tag{10}$$

$$y_t = A_t F\left(k_t\ ,\ N_t\right). \tag{11}$$

The level of technology is assumed to follow a statonary AR(1) process that can be written

$$\log\left(A_t\right) = \rho \log\left(A_{t-1}\right) + \epsilon_t, \qquad 0 < \rho < 1, \tag{12}$$

where lower-case variables are normalized by X_t.

The competitive equilibrium to this economy can be found as the solution to a social planning problem in which the planner maximizes the utility of the representative agent subject to the economy's resource constraint and to the capital accumulation equation. The first order conditions are

$$u_1\left(c_t\ ,\ L_t\right) = \lambda_t; \tag{13}$$

$$u_2\left(c_t\ ,\ L_t\right) = \omega_t = w_t \lambda_t; \tag{14}$$

$$\lambda_t A_t F_2\left(k_t\ ,\ N_t\right) = \omega_t; \tag{15}$$

$$\gamma\lambda_t = \beta E_t \left(\lambda_{t+1} A_{t+1} F_1 \left(k_{t+1}\ ,\ N_{t+1}\right) + 1 - \delta\right). \tag{16}$$

The variable ω_t is the marginal value of leisure (the units are utils per hour) and is associated with the constraint $L_t + N_t = 1$. It can be rewritten as $w_t\lambda_t$, where w_t can be thought of as the competitive real wage rate (whose units are units of output per hour, whereas λ_t is measured in utils per unit of output). This is the basic system to be linearized. To simplify, let's use the following functional form for the utility function:

$$u\left(c_t\ ,\ L_t\right) = \log\left(c_t\right) + \frac{\theta}{1-\eta}\left(\left(1 - N_t\right)^{(1-\eta)} - 1\right) \tag{17}$$

Linearizing the first order condition (13) gives

$$\begin{aligned}
\frac{1}{c} - \frac{1}{c^2}\left(c_t - c\right) &\approx \lambda + \left(\lambda_t - \lambda\right) \\
\Rightarrow -\frac{c_t - c}{c} &\approx \frac{\lambda_t - \lambda}{\lambda} \\
\Rightarrow \hat{c}_t &= -\hat{\lambda}_t,
\end{aligned} \tag{18}$$

where variables without time subscripts denote their steady-state levels and variables with hats are measured in proportional deviations from their steady-state values, that is to say for a variable x_t

$$\hat{x}_t \equiv \frac{x_t - x}{x} \approx \log\left(x_t / x\right).$$

It is often very convenient and succinct to linearize in terms of proportional deviations from steady-state values. Many equations in DSGE models are close to log-linear, which leads to simple expressions when they are linearized in this way.

The second first order condition (14) can be written as

$$\theta {L_t}^{-\eta} = \lambda_t w_t. \tag{19}$$

Linearizing this equation gives

$$\theta L^{-\eta} - \eta\theta L^{\eta-1}\left(L_t - L\right) = w\lambda + \lambda\left(w_t - w\right) + w\left(\lambda_t - \lambda\right)$$

$$\Rightarrow -\eta\hat{L}_t = \hat{w}_t + \hat{\lambda}_t \tag{20}$$

With a Cobb-Douglas production function,

$$y_t = A_t {k_t}^{(1-\alpha)} {N_t}^{\alpha}, \tag{21}$$

we have

$$w_t = A_t\left(k_t/N_t\right)^{(1-\alpha)}.$$

This equation is perfectly log-linear and becomes

$$\hat{w}_t = \hat{A}_t + (1-\alpha)\left(\hat{k}_t - \hat{N}_t\right). \tag{22}$$

The constraint on the representative agent's time gives

$$N_t - N + L_t - L = 0$$

$$\Rightarrow N\frac{N_t - N}{N} + L\frac{L_t - L}{L} = 0$$

$$\hat{N}_t + \hat{L}_t = 0. \tag{23}$$

The economy's resource constraint can be written as

$$c_t + i_t = y_t = A_t {k_t}^{(1-\alpha)} {N_t}^{\alpha}$$

$$\Rightarrow c_t - c + i_t - i = y_t - y$$

$$\Rightarrow \left(\frac{c}{y}\right)\frac{c_t - c}{c} + \left(\frac{i}{y}\right)\frac{i_t - i}{i} = \frac{y_t - y}{y}$$

$$\Rightarrow \left(\frac{c}{y}\right)\hat{c}_t + \left(\frac{i}{y}\right)\hat{i}_t = \hat{y}_t \tag{24}$$

and

$$\hat{y}_t \approx \log\left(y_t/y\right) = \log\left(A_t/A\right) + (1-\alpha)\log\left(k_t/k\right) + \alpha\log\left(N_t/N\right)$$

$$\Rightarrow \left(\frac{c}{y}\right)\hat{c}_t + \left(\frac{i}{y}\right)\hat{i}_t = \hat{A}_t + (1-\alpha)\hat{k}_t + \alpha\hat{N}_t. \tag{25}$$

The fourth first order condition (16) is a little harder to loglinearize. Taking total differentials of the equation gives

$$\gamma d\lambda_t =$$

$$E_t\Big\{\beta A(1-\alpha)N^{\alpha}k^{-\alpha}d\lambda_{t+1} + \beta\lambda(1-\alpha)N^{\alpha}k^{-\alpha}dA_{t+1}$$

$$+\beta\lambda A(1-\alpha)k^{-\alpha}\alpha N^{(\alpha-1)}dN_{t+1} + \beta\lambda A(1-\alpha)N^{\alpha}(-\alpha)k^{(-\alpha-1)}dk_{t+1}\Big\}.$$

Using the fact that in the steady state

$$\gamma\lambda = \beta\lambda A N^{\alpha}k^{-\alpha}$$

this simplifies to

$$\frac{d\lambda_t}{\lambda} = E_t\frac{d\lambda_{t+1}}{\lambda} + E_t\frac{dA_{t+1}}{A} + \alpha E_t\left(\frac{dN_{t+1}}{N} - \frac{dk_{t+1}}{k}\right).$$

Using

$$\frac{dx}{x} = \frac{x_t - x}{x}$$

for any variable x gives

$$\hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + E_t\hat{A}_{t+1} + \alpha E_t\hat{N}_{t+1} - \alpha\hat{k}_{t+1}. \tag{26}$$

The stochastic process for technology is already log-linear and gives

$$\hat{A}_t = \rho \hat{A}_{t-1} + \epsilon_t. \tag{27}$$

Finally, taking total differentials of the equation for the accumulation of capital gives

$$\gamma dk_{t+1} = (1-\delta)dk_t + di_t$$

$$\Rightarrow \gamma \frac{dk_{t+1}}{k} = (1-\delta)\frac{dk_t}{k} + \frac{i}{k}\frac{di_t}{i}.$$

Using the fact that in the steady state

$$\gamma k = (1-\delta)k + i \Rightarrow \frac{i}{k} = \gamma + \delta - 1$$

we get

$$\gamma \hat{k}_{t+1} = (1-\delta)\hat{k}_t + (\gamma + \delta - 1)\hat{i}_t. \tag{28}$$

Equations (18), (20), (22), (23), (25), (26), (27), and (28) constitute a system of eight equations. The unknowns are $\hat{\lambda}_t$, $\hat{c}_t$, $\hat{w}_t$, $\hat{k}_t$, $\hat{N}_t$, $\hat{L}_t$, $\hat{A}_t$ and $\hat{i}_t$. Of these variables, $\hat{\lambda}_t$, $\hat{k}_t$ and $\hat{N}_t$ appear in the system dated at time t and at time $t+1$. This is precisely the problem alluded to earlier that $\hat{N}_t$ is not a truly dynamic variable.

Two (at least) options are available to reduce the system to a minimal state space representation. The first option is to eliminate all variables other than k_t and λ_t algebraically. This leads to the following system:

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} \begin{bmatrix} E_t\hat{\lambda}_{t+1} \\ \hat{k}_{t+1} \\ \hat{A}_t \end{bmatrix} =$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_t \\ \hat{k}_t \\ \hat{A}_{t-1} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \epsilon_t. \tag{29}$$

The values of the elements are given in an appendix.

It is apparent that even for a model this simple, reducing it algebraically to a system of the form (29) is fairly tedious. In models with "bells and whistles" such as adjustment costs, habit formation, nominal rigidities, etc., this will become extremely tedious, and so complicated that the probability of an algebraic eror bcomes quite high.

A second option is to write the system as two subsystems:

$$G1 \begin{bmatrix} \hat{A}_t \\ \hat{k}_{t+1} \\ E_t\hat{\lambda}_{t+1} \end{bmatrix} + G2 \begin{bmatrix} \hat{A}_{t-1} \\ \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} + G3 \begin{bmatrix} \hat{c}_t \\ \hat{L}_t \\ \hat{N}_t \\ \hat{i}_t \\ \hat{w}_t \end{bmatrix} + G4\epsilon_t = 0,$$

$$H1 \begin{bmatrix} \hat{c}_t \\ \hat{L}_t \\ \hat{N}_t \\ \hat{i}_t \\ \hat{w}_t \end{bmatrix} + H2 \begin{bmatrix} \hat{A}_t \\ \hat{k}_{t+1} \\ E_t\hat{\lambda}_{t+1} \end{bmatrix} + H3 \begin{bmatrix} \hat{A}_{t-1} \\ \hat{k}_t \\ \hat{\lambda}_t \end{bmatrix} = 0.$$

We have still eliminated $E_t\hat{N}_{t+1}$ from the model by substitution. This system can be solved numerically to yield an equation system of the form (29).

3.1.1 Calibration

[To be added.]

3.2 Application to the Canonical New Keynesian Model

See Ambler (2010) for more details on the canonical model and its solution as a system of linear difference equations.

[To be completed.]

4 Basic Procedure

We assume the system can be written in state space form as:

$$\begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t, \tag{30}$$

where x_t is a vector of n predetermined variables, y_t is a vector of m non-predetermined variables, ${}_t y_{t+1}$ is the conditinal expectation of y_{t+1} based on information available at time t and z_t is a vector of k forcing variables, the D_i are conformable submatrices of the matrix D, and the A_{ij} are conformable submatrices of the matrix A. For stochastic simulations, z_t will normally contain white noise shocks. The dynamics associated with the forcing variables (autoregressivie or moving average components) will be incorporated in the A matrix. We initially assume that saddlepoint stability holds, so that the predetermined states are associated with stable eigenvalues of A (whose absolute value is less than 1), and the non-predetermined states are associated with unstable eigenvalues (absolute value greater than 1). We relax this assumption below.

Farmer (1999) has a detailed discussion of linear rational expectations models in which there are more stable eigenvalues than predetermined variables. Such models have multiple dynamic equilibria, some of which may depend on extraneous random variables, otherwise known as "sunspot equilibria." It is also possible to consider models in which one or more eigenvalues associated with the predetermined state variables

are exactly equal to one. These models do not have unique steady states: temporary shocks will have permanent effects on one or more of the model's state variables. For this reason, one cannot talk about linearizing around "the" steady state. In stochastic simulations with long time horizons, one or more of the state variables will eventually be driven far away from their initial values, in which case the linear approximation is likely to become highly inexact. Because of these considerations, I only consider the standard case where the stable eigenvalues are strictly less than one in absolute value.

To arrive at this state-space form of the model, it is necessary to eliminate "non-dynamic" endogenous variables from the system. These endogenous variables are often of economic interest: output, employment, consumption, etc. We will reintroduce these variables below in the context of the stochastic properties of the model (see equation (38). Sometimes it is a bit hard to see that a particular variable in a model is not really a dynamic or state variable, since it appears in different equations with different time subscripts. For example, in the standard RBC model considered by King, Plosser and Rebelo (1988) and others, the first order condition for the representative agent's (or the social planner's) choice of labor supply looks like

$$\frac{\partial U}{\partial N_t} + \frac{\partial U}{\partial C_t} W_t = 0,$$

where C_t is consumption, N_t is hours worked and W_t is the real wage, so that the marginal disutility of work is equal to the marginal utility of the consumption provided by working a little bit more. In this equation, hours worked are dated t. The consumption Euler equation in such models looks like

$$\frac{\partial U}{\partial C_t} - \beta E_t \left[\frac{\partial U}{\partial C_{t+1}} (R_{t+1} + (1 - \delta)) \right] = 0,$$

where R_{t+1} is the real rental rate of capital at $t + 1$. In a competitive model, the real rental rate of capital will depend on the capital/labor ratio, and hence R_{t+1} depends on

N_{t+1}, so that both current and future hours appear in the model. This does *not* mean that N_t is a truly dynamic (state) variable that should appear in the y_t vector in equation (30). We can lead the first order condition for hours by one period to get

$$\frac{\partial U}{\partial N_{t+1}} + \frac{\partial U}{\partial C_{t+1}} W_{t+1} = 0.$$

We can then substitute out W_{t+1} using the first order condition for firms' profit maximization and express N_{t+1} as a function of C_{t+1}, K_{t+1} (the capital stock), and Z_{t+1} (the Solow residual). Typically, these variables are in fact true state variables of the model. We can then substitute out both N_t and N_{t+1} from the model. N_t is an endogenous variable that is related statically to the state variables of the model. For more details on what can be called the "minimal state space representation" of dynamic models, see McCallum (1999).

Alternatively, we could develop an algorithm that automatically takes care of variables such as N_t in the previous example. Such variables will be associated with zero roots or eigenvalues if they are not eliminated from the system of equations that characterize the model's equilibrium. They typically cannot be written in the form given in equation (30). Instead, we would have a system of the form

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t, \qquad (31)$$

where the B matrix is not necessarily invertible. See King and Watson (1998) and Klein (2000). Their algorithms rely on the Schur decomposition, which is more general than the QR decomposition used to calculate the eigenvalues of the A matrix in the basic algorithm described below. We will consider solving models this way below, but for now we stick to models in which extraneous

Note that in the basic growth model considered by King, Plosser and Rebelo, the x_t and y_t vectors are both in fact scalars. The predetermined state variable in their model

is the capital stock. The costate variable is the Lagrange multiplier associated with the economy's global resource constraint. It is usually possible to write down the planning problem so that each state variable is associated with a costate variable. However, this is usually not the *only* way of writing down the model. One should not automatically assume that the number of elements in x_t is equal to the number of elements in y_t.

We know that the relationship between the eigenvectors and eigenvalues of the A matrix is given by

$$c_i A \equiv \lambda_i c_i,$$

where c_i is the i^{th} *left* eigenvector of the matrix A (of dimension 1 by $(n+m)$) and λ_i is the i$^{\text{th}}$ eigenvalue. Stacking the eigenvectors together, we have

$$CA = JC,$$

where C is the $(n+m) \cdot (n+m)$ matrix whose rows consist of the left eigenvectors of A, and where J is the diagonal matrix whose diagonal elements consist of the corresponding eigenvalues of A. I assume that the eigenvalues are ordered so that the m unstable eigenvalues appear after the n stable eigenvalues.[1] The eigenvectors must be ordered so that each one is associated with its corresponding eigenvalue. It is also extremely important that the eigenvalues are sorted so that the unstable eigenvalues are associated with (in the same row as) the non-predetermined state variables.

Note that the algebra that I have developed here makes use of *left* eigenvectors. These are not necessarily the same as *right* eigenvectors, which satisfy the equation

$$A\tilde{c}_i \equiv \tilde{c}_i \lambda_i.$$

Some canned computer algorithms (the *GAUSS* command $\text{eigv}(\cdot)$ and the *Matlab* com-

[1] Most computer matrix languages like *GAUSS* contain routines that automatically calculate eigenvalues and eigenvectors and routines that sort the rows or columns of a matrix into ascending or descending order, so that it is relatively trivial to generate the C and J matrices numerically given the matrix A.

mand eig$(\cdot)$ are two examples) automatically calculate right eigenvectors. In order to calculate left eigenvectors using *GAUSS* or *Matlab*, we must first **transpose** the A matrix. Also, since *GAUSS* and *Matlab* do not necessarily give us the eigenvectors in order of ascending absolute values, we must perform a simultaneous sort on the eigenvalues **and corresponding eigenvectors** (see the section below on complex eigenvalues for additional remarks).

The system can be rewritten as

$$\begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} = C^{-1} J C \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t$$

$$\Rightarrow C \begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} = J C \begin{bmatrix} x_t \\ y_t \end{bmatrix} + C \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t$$

or

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} =$$

$$J \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t,$$

where the C_{ij} are conformable submatrices of the C matrix. Write this as

$$\begin{bmatrix} \tilde{x}_{t+1} \\ {}_t \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} z_t, \tag{32}$$

where

$$\tilde{x}_t \equiv C_{11} x_t + C_{12} y_t,$$

$$\tilde{y}_t \equiv C_{21} x_t + C_{22} y_t,$$

$$G_1 \equiv C_{11} D_1 + C_{12} D_2,$$

$$G_2 = C_{21}D_1 + C_{22}D_2,$$

and where the J_i are conformable submatrices of J. We have succeeded in *decoupling* the equation system. We have $(n + m)$ scalar first difference equations, with the non-predetermined variables depending only on the unstable eigenvalues of A.

We can now solve for $\tilde{y}_t$ simply by *iterating forward*, as follows:

$$_t\tilde{y}_{t+1} = J_2\tilde{y}_t + G_2 z_t$$

$$\Rightarrow \tilde{y}_t = J_2^{-1}\ {}_t\tilde{y}_{t+1} - J_2^{-1}G_2 z_t.$$

Forwarding the last equation and taking expectations at time t gives

$$_t\tilde{y}_{t+1} = J_2^{-1}\ {}_t\tilde{y}_{t+2} - J_2^{-1}G_2\ {}_t z_{t+1}.$$

Substituting back in the previous equation, we get

$$\tilde{y}_t = J_2^{-1}\left(J_2^{-1}\ {}_t\tilde{y}_{t+2} - J_2^{-1}G_2\ {}_t z_{t+1}\right)\ - J_2^{-1}G_2 z_t.$$

We can continue, substituting out $_t\tilde{y}_{t+i}$ in the same manner, to obtain

$$\tilde{y}_t = -\sum_{i=0}^{\infty} J_2^{-(i+1)}G_2\ {}_t\tilde{z}_{t+i}. \tag{33}$$

The current value of $\tilde{y}_t$ thus depends on the entire sequence of future expected values of the forcing variables. This technique can be used to simulate *perfect foresight models*, in which the entire future paths of the forcing variables are assumed to be known with certainty. We must in general impose that the infinite sum in the previous equation is convergent. In practice, we must typically assume that the values of the forcing variables converge to constants in a finit number of periods, so that we can find an analytical solution for the infinite sum of terms. Concretely, this means that the forcing

variables $\tilde{z}_{t+i}$ must grow more slowly than the inverses of the unstable eigenvalues $J_2^{-(i+1)}$ shrink.

In the special case (relevant for stochastic simulations) where the z_t are white noise shocks, we have

$$\tilde{y}_t = -J_2^{-1} G_2 z_t. \tag{34}$$

In stochastic simulations, the elements of z_t are nearly always white noise shocks. In cases where the exogenous variables of the model follow autoregressive or even vector autoregressive processes, the state vector of the model is augmented with current or lagged values of these exogenous variables. Higher order autoregressive processes can always be transformed into a system of equations in first-order processes.[2] Using the definition of $\tilde{y}_t$ given above, we have

$$C_{21} x_t + C_{22} y_t = -J_2^{-1} G_2 z_t$$

$$\Rightarrow y_t = -C_{22}^{-1} C_{21} x_t - C_{22}^{-1} J_2^{-1} G_2 z_t.$$

Substituting out G_2 using the definition given above, we have

$$y_t = -C_{22}^{-1} C_{21} x_t - C_{22}^{-1} J_2^{-1} \left(C_{21} D_1 + C_{22} D_2 \right) z_t. \tag{35}$$

We have succeeded in expressing y_t in terms of *current* values of the predetermined states and the forcing variables. We then have

$$x_{t+1} = A_{11} x_t + A_{12} y_t + D_1 z_t. \tag{36}$$

So, by transforming and then retransforming the system, the equations for x_t and y_t become *recursive*. We can easily set up a program to simulate this system numerically by using the following pseudo-algorithm.

[2] See Harvey (1990) for details.

4.1 Pseudo-algorithm for Numerical Simulations

- Generate random shocks z_i for $1 \leq i \leq T$, using a pseudo-random number generator.
- Set up matrices with T rows to store results.
- Initialize x_0, y_0, z_0. Most often, this involves setting these variables equal to their values in a deterministic steady state. If the variables have been transformed so that they measure deviations or proportional deviations from a deterministice steady state, the initial values will typically be zero.
- Set $i = 1$.
- Do from $i = 1$ to $i = T$.
- Set $x_i = A_{11}x_{i-1} + A_{12}y_{i-1} + D_1 z_{i-1}$.
- Set $y_i = -C_{22}^{-1}C_{21}x_i - C_{22}^{-1}J_2^{-1}\left(C_{21}D_1 + C_{22}D_2\right)z_i$.
- Store results.
- Set x_{i-1} equal to x_i.
- Set y_{i-1} equal to y_i.
- Set i equal to $i + 1$.
- End do.

4.2 Note Concerning Complex Eigenvalues

Note that the A matrix may contain complex eigenvalues. Nothing in principle is wrong with this. A dynamic system with complex eigenvalues will (if it is stable) display **damped oscillations** in response to any kind of exogenous shock. However, we would certainly be worried if the output of our algorithm included anything other than real

values. The results of multiplying together the (possibly) complex elements of the matrices in equation (1) above must always be real. This is actually one way of checking that our algorithm makes sense (and that we have not made any programming errors). Due to numerical errors, the imaginary parts of the y_t vector will typically not be exactly zero in the presence of complex eigenvalues. However, they should be very small. If not, we know that the program has gone wrong somewhere. A common reason for getting nonzero imaginary parts is not sorting the eigenvectors along with the eigenvalues when we sort the latter into ascending absolute values after using a canned package to numerically calculate the eigenvalues of the A matrix.

Previous versions of *GAUSS* and *Matlab* could not easily handle complex-valued variables, but in the current versions variables are complex by default. Rather than having to program around this problem, we can simply ignore it and assume that *GAUSS* and *Matlab* can handle the problem automatically. In previous versions, it was possible to use the trick (see Hirsch and Smale, 1974) that if we substitute augmented real matrices of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

for the complex matrices $A + Bi$, then all ordinary matrix operations such as matrix inversion and matrix multiplication on the augmented matrices preserve the real and imaginary parts.

5 Stochastic Properties

In the case that the z_t matrix is a matrix of stochastic white-noise shocks, this methodology can easily be extended to the calculation of the asymptotic variance-covariance properties of the model's state variables.[3] This is of interest when we want an easy

[3] If the exogenous variables of the model follow ARMA processes, it is always possible to write the state-space form of the model in terms of white noise shocks by a suitable choice of state variables. See Harvey (1990) for details.

way to calculate the model's predictions concerning comovements. We can do so without having to resort to stochastic simulations. This is also of interest when we wish to *estimate* or *test* the model using methods based on the generalized method of moments (*GMM*). These predictions are *exact* in the sense that they do not depend on a particular random draw of stochastic shocks used to simulate a finite sample used to calculate the comovements. They are less than exact in the sense that they depend on a linear approximation to the true model. Ambler, Guay and Phaneuf (2010) use similar techniques to estimate a business cycle model with nominal wage contracts and labor adjstment costs, and to test the model's overidentifying restrictions.

The model generates data measured as proportional deviations from a steady state. We can recover the variables in levels by renormalizing them. We can also typically easily recover the variables in levels when the model has been stationarized to get rid of a deterministic or stochastic trend. Finally, we can fairly easily cope with measuring the model's variables in growth rates, by using suitable algebraic transformations. We would use such transformations so that the model's variables are measured in the same way as the variables in the data set being used to estimate and test the model. In cases where the model is to be estimated using filtered data (using, for example, the Hodrick-Prescott filter), the model must be simulated, since the filter typically depends on the sample size.

5.1 Unconditional Variance-Covariance Matrix of the State Variables

The conditional variance of any I(1) variable must necessarily be finite. However, the unconditional variance will always be undefined. It is therefore necessary to eliminate all I(1) variables from the solution before calculating variances and autocovariances.

The solution for the state variables will always be of the form

$$x_{t+1} = Fx_t + Gz_t.$$

Let's say that we want to eliminate the k^{th} variable from the solution. Call I^k the $n \times n$ identity matrix with its k^{th} row knocked out. Premultiplying by I^k gives

$$I^k x_{t+1} = I^k F x_t + I^k G z_t.$$

It is obvious that this technique can be applied to eliminate more than one variable from the solution. Note that this means that none of the other variables in the system can depend on the k^{th} variable, since we also get rid of $x_{k,t}$ which is on the right hand side of the equation. This is natural. If only the stationary variables in x_t remain, their solution cannot depend on nonstationary variables, or they themselve would be nonstationary. There could only be a problem if some of the nonstationary variables in x_t were cointegrated, and the solution for one of the stationary variables depended on the exact cointegrating vector. Unless the cointegrating relationship is imposed by the modeler, this would be a highly unlikely coincidence.

We make use of the fact that, with suitable algebraic transformations, the dynamics of the system can be written as a first-order vector autoregression, for which the covariance properties are well known. Substituting in the solution for the forward-looking state variables, we have

$$x_{t+1} = A_{11}x_t + A_{12}y_t + D_1 z_t.$$

$$= A_{11}x_t$$

$$+A_{12}\left(-C_{22}^{-1}C_{21}x_t - C_{22}^{-1}J_2^{-1}\left(C_{21}D_1 + C_{22}D_2\right)z_t\right)$$

$$+D_1 z_t,$$

which gives an equation of the form

$$x_{t+1} = Fx_t + Gz_t. \tag{37}$$

We can now use standard techniques (see Hamilton, 1994, p.265) in order to calculate the variance-covariance matrix of the predetermined state variables as a function of the variance-covariance matrix of the exogenous shocks. We assume that the shocks z_t are distributed as

$$z_t \sim (0, \Sigma)$$

and are serially uncorrelated. Following Hamilton, we have

$$\text{vec}(\Omega) \equiv \text{vec}(\text{var}(x)) = (I - F \otimes F)^{-1} \text{vec}(G\Sigma G'),$$

where $\otimes$ is the Kronecker product operator, and vec is the vectorization operator, which stacks the columns of a matrix one after the other in a long column vector.

5.2 Variance-Covariance Matrix of Non-State Endogenous Variables

Often, we are not interested in the stochastic properties of the state variables themselves, but rather in the properties of a set of nonstate endogenous variables that have been substituted out of the model in order to arrive at its state space form. We have a relationship of the following form between the predetermined state variables and the nonstate endogenous variables of the model:[4]

$$v_t = Hx_t + Jz_t. \tag{38}$$

[4]Of course, these equations may have to be linearized along with the dynamic equations of the model.

It is then straightforward to show that the variance-covariance matrix of the endogenous variables is given by

$$\text{var}(v) = H\Omega H' + J\Sigma J'.$$

5.3 Autocovariances

Particularly when we are interested in the dynamic properties of the models, including the persistence of its state and/or endogenous variables, we need to calculate the autocovariance function of the endogenous variables. This is easily done by postmultiplying (38) above and taking expectations, as follows:

$$v_t v'_{t-1} = (Hx_t + Jz_t)\left(x'_{t-1}H' + z'_{t-1}J'\right)$$

$$\Rightarrow \text{Autocov}\left(vv_{-1}\right) = H\text{Autocov}\left(xx_{-1}\right)H' + HG\Sigma J'$$

Similarly,

$$\text{Autocov}\left(vv_{-2}\right) = H\text{Autocov}\left(xx_{-2}\right)H' + HFG\Sigma J',$$

$$\text{Autocov}\left(vv_{-3}\right) = H\text{Autocov}\left(xx_{-3}\right)H' + HF^2G\Sigma J',$$

etc. The i^{th} order autocovariance matrix of the predetermined state variables is given by

$$\text{Autocov}\left(xx_{-i}\right) = F\text{Autocov}\left(xx_{-(i-1)}\right). \tag{39}$$

When $i = 0$, this is just the covariance matrix of the predetermined states.

5.4 A Note on Measuring in Growth Rates

Subsequent to the work of Cogley and Nason (1995), it is often customary to evaluate dynamic general equilibrium business cycle models by calculating series in growth rates. If we measure the data in growth rates, for reasons of compatibility it is normal

to generate the predictions of the model in terms of growth rates. As it stands, because of the fact that we linearize around the steady state, all our variables are measured as proportional deviations from the steady state. Trending variables are also detrended through a normalization that involves dividing by a variable that is cointegrated with them.

The procedure involves "denormalizing" the normalized variables (multiplying by the variable originally used to normalize them), and then transforming the variables of interest into first differences. For example, assume that normalized output equals output divided by the current level of technology from the aggregate production function, so that we have

$$y_t = A_t N_t^{\alpha} K_t^{1-\alpha},$$

with

$$\ln(A_t) = \ln(A_{t-1}) + \varepsilon_t$$

and

$$\bar{y}_t \equiv y_t / A_t.$$

Then, if we wish to generate the model's predictions in growth rates, we have

$$\ln(y_t) - \ln(y_{t-1}) = \ln(\bar{y}_t) - \ln(\bar{y}_{t-1}) + \varepsilon_t.$$

The rate of growth of normalized output is approximately equal to the first difference of the proportional deviation of normalized output from its steady state. So, in order to generate predictions in growth rates, we must be able to measure the model's variables in first differences. We have

$$(v_t - v_{t-1}) = H(x_t - x_{t-1}) + J(z_t - z_{t-1}).$$

We can augment the state space form of the model as follows:

$$\begin{bmatrix} x_{t+1} \\ (x_{t+1} - x_t) \\ z_t \end{bmatrix} = \begin{bmatrix} F & 0 & 0 \\ 0 & F & -G \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ (x_t - x_{t-1}) \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} G \\ G \\ I \end{bmatrix} z_t.$$

Once we transform the model in this manner,[5] all of the derivations of the preceding subsections go through without modification, including the algebra for calculating the variance-covariance matrices of state variables and endogenous variables and the autocovariance matrices. This formulation allows us also to measure some variables in first differences and others in levels. This may be useful in cases where there are theoretical reasons to believe that some variables remain stationary while others contain stochastic trends. This would be the case, for example, of the neoclassical growth model with technology following a random walk with drift as above. If the representative agent's utility function obeys certain restrictions which are compatible with the existence of balanced growth paths,[6] per capita hours remain stationary while output and the components of aggregate demand contain a common trend.

6 Conditional Variances and Autocovariances

In the case of conditional variances and autocovariances, they can be calculated for any finite horizon without [To be added.]

[5] Note that since the shocks are assumed to be white noise, the eigenvalues associated with the shocks in the transformed state transition matrix will all be zero. This poses no technical difficulty, since zero eigenvalues are stable and are associated with the z_{t-1}, which are predetermined state variables.

[6] See King, Plosser and Rebelo (1988) for a discussion.

7 Predictions and Prediction Errors

It is possible to generate predictions using the solution developed here, and to decompose forecast errors according to which

[To be completed.]

8 Impulse Response Functions

It is straightforward to simulate the model in response to a shock that lasts only one period instead of a series of shocks generated by a random number generator. In the first step of the pseudo-algorithm detailed above, just substitute a vector of zeros except for the first element of the shock to be simulated.

Generalization

In this section, we extend the algorithm to the case where the system can be written as:

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ {}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} z_t, \tag{40}$$

where the B matrix is not necessarily invertible. This means that the standard Blanchard and Kahn algorithm cannot be applied.

We make use of the *generalized Schur decomposition* or *QZ decomposition*. See Klein (2000) and Heer and Maussner (2009) for more details. There is a theorem that says that we can factorize the A and B matrices as follows:

$$U^\dagger B V = S,$$

$$U^\dagger A V = T, \tag{41}$$

where U and V are *unitary matrices* (square complex matrices such that $U^\dagger U = UU^\dagger = I$, with $U^\dagger$ the complex conjugate matrix of the matrix U) and both S and T are upper-triangular. The matrices can be arranged so that the eigenvalues of the linear matrix pencil $A - \lambda B$ are in increasing absolute value. Define the transformed variables

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix}. \tag{42}$$

The system can be written

$$U^\dagger BV \begin{bmatrix} \tilde{x}_{t+1} \\ {}_t\tilde{y}_{t+1} \end{bmatrix} = U^\dagger AV \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \end{bmatrix} + U^\dagger D z_t,$$

or

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{t+1} \\ {}_t\tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \tilde{D} z_t. \tag{43}$$

First of all, consider the case where the z_t are i.i.d. shocks. The lower part of equation (43) can be written as follows

$$S_{22\,t}\tilde{y}_{t+1} = T_{22}\tilde{y}_t + \tilde{D} z_t. \tag{44}$$

We have

$$\tilde{y}_t = {T_{22}}^{-1} S_{22\,t}\tilde{y}_{t+1} - {T_{22}}^{-1}\tilde{D} z_t$$

$$\Rightarrow_t \tilde{y}_{t+i} = {T_{22}}^{-1} S_{22\,t}\tilde{y}_{t+i+1}, \quad i > 0,$$

$$\Rightarrow \tilde{y}_t = -{T_{22}}^{-1}\tilde{D} z_t. \tag{45}$$

Now consider the more complicated case where z_t follows the stochastic process given by

$$z_t = \Pi z_{t-1} + \epsilon_t. \tag{46}$$

In this case, we can use a variation of the undetermined coefficients method to fine the solution for $\tilde{y}_t$ as a function of z_t. Posit a solution of the form

$$\tilde{y}_t = \Phi z_t. \tag{47}$$

The last equation of (44) can be written as

$$S_{n(y),n(y)_t}\tilde{y}_{n(y),t+1} = T_{n(y),n(y)}\tilde{y}_{n(y),t} + d'_{n(y)}z_t$$

and the last line of (47) can be written as

$$\tilde{y}_{n(y),t} = \phi'_{n(y)}z_t.$$

Using the posited stochastic process for z_t and substituting, we get

$$\left[S_{n(y),n(y)}\phi'_{n(y)}\Pi - T_{n(y),n(y)}\phi'_{n(y)} - d'_{n(y)}\right] z_t = 0$$

$$\Rightarrow \phi'_{n(y)} = d'_{n(y)}\left(S_{n(y),n(y)}\Pi - T_{n(y),n(y)}I_{n(z)}\right)^{-1}.$$

Now consider the next-to-last line of (44). This can be written as

$$S_{n(y)-1,n(y)-1_t}\tilde{y}_{n(y)-1,t+1} + S_{n(y)-1,n(y)_t}\tilde{y}_{n(y),t+1}$$

$$= T_{n(y)-1,n(y)-1}\tilde{y}_{n(y)-1,t}T_{n(y)-1,n(y)}\tilde{y}_{n(y),t} + d'_{n(y)-1}z_t.$$

Using the same procedure as for $\phi'_{n(y)}$, we get

$$\phi'_{n(y)-1} = \left(T_{n(y)-1,n(y)}\phi'_{n(y)} - S_{n(y)-1,n(y)}\phi'_{n(y)} + d'_{n(y)-1}\right)$$

$$\times \left(T_{n(y)-1,n(y)-1} + S_{n(y)-1,n(y)-1}\Pi\right)^{-1}.$$

The pattern is clear. We can solve recursively for the elements of Φ, and the solution can be written as

$$\phi_i' = \left[d_i' + \sum_{j=i+1}^{n(y)} \left(T_{i,i}\phi_j' - S_{i,j}\phi_j'\Pi \right) \right] \left(S_{i,i}\Pi - T_{i,i}I_{n(z)} \right)^{-1} \tag{48}$$

Given the solution for Φ, we need to write down the explicit solution for y_t and x_t. Using the lower part of (42) we have

$$y_t = V_{21}\tilde{x}_t + V_{22}\tilde{y}_t.$$

The upper part can be solved for $\tilde{x}_t$, which gives

$$\tilde{x}_t = {V_{11}}^{-1}x_t - {V_{11}}^{-1}V_{12}\tilde{y}_t.$$

Substituting, we get

$$y_t = \left(V_{22} - V_{21}{V_{11}}^{-1}V_{12} \right) \tilde{y}_t + V_{21}{V_{11}}^{-1}x_t$$

$$y_t = V_{21}{V_{11}}^{-1}x_t + \left(V_{22} - V_{21}{V_{11}}^{-1}V_{12} \right) \Pi z_t. \tag{49}$$

The non-predetermined variables are therefore functions of the predetermined variables and the exogenous variables in equilibrium.

Finally, solve for the equilibrium law of motion for x_t. Using the top part of (43), we get

$$S_{11}\tilde{x}_{t+1} = -S_{12t}\tilde{y}_{t+1} + T_{11}\tilde{x}_t + T_{12}y_t + \tilde{D}_1 z_t$$

$$= -S_{12}\Phi\Pi z_t + T_{11}\tilde{x}_t + T_{12}\Phi z_t + \tilde{D}_1 z_t.$$

The S_{11} matrix is invertible since it has only stable eigenvalues. So we get

$$\tilde{x}_{t+1} = {S_{11}}^{-1} T_{11} \tilde{x}_t + {S_{11}}^{-1} \left(T_{12} + \tilde{D}_1 - S_{12} \Phi \Pi \right) z_t.$$

From the upper part of (42), we have

$$\tilde{x}_t = {V_{11}}^{-1} x_t - {V_{11}}^{-1} V_{12} \tilde{y}_t$$

$$= {V_{11}}^{-1} x_t - {V_{11}}^{-1} V_{12} \Phi z_t$$

and

$$\tilde{x}_{t+1} = {V_{11}}^{-1} x_{t+1} - {V_{11}}^{-1} V_{12t} \tilde{y}_{t+1}$$

$$= {V_{11}}^{-1} x_{t+1} - {V_{11}}^{-1} V_{12} \Phi \Pi z_t.$$

Substituting and solving we get

$$x_{t+1} = V_{11} {S_{11}}^{-1} T_{11} {V_{11}}^{-1} x_t +$$

$$\left(V_{12} \Phi \Pi - V_{11} {S_{11}}^{-1} T_{11} {V_{11}}^{-1} V_{12} \Phi V_{11} {S_{11}}^{-1} \left(T_{12} + \tilde{D}_1 - S_{12} \Phi \Pi \right) \right) z_t. \quad (50)$$

The solutions for x_t and y_t are of the same functional form as in the Blanchard-Kahn case. All the derivations concerning the stochastic properties of the model go through without any changes.

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Appendix

We list here the values of the elements of the matrices in equation (29) above.

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1-\alpha\left((1-\alpha)+\eta\right)^{-1} & \left(\frac{i}{y}\right)\frac{\gamma}{\gamma+\delta-1} & 0 \\ \rho+\rho\alpha\left((1-\alpha)+\eta\right)^{-1} & \alpha(1-\alpha)\left((1-\alpha)+\eta\right)^{-1} & 1+\alpha\left((1-\alpha)+\eta\right)^{-1} \end{bmatrix}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \rho & 0 & 0 \\ 0 & \left(\frac{i}{y}\right)\frac{1-\delta}{\gamma+\delta-1} & \left(\frac{c}{y}\right)+\alpha\left((1-\alpha)+\eta\right)^{-1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$