ECO511 Eric Sims' RBC Model

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Introduction

- Why study Sims' model?
  1. We can solve the model analytically for the steady state. This is not often the case with RBC models.
  2. We can look explicitly at how to use properties of the steady state to calibrate the model's parameter values.
  3. We can invert this process to examine how changes in the parameter values of the model affect its steady state.
  4. We can calculate a linearized version by hand and do a qualitative analysis of the model's dynamics using phase diagrams.
  5. The linearized dynamic conditions tell us the slopes of "isoclines" (curves along which either consumption is stationary or the capital stock is stationary), and what is the direction of change of these variables on either side of these isoclines.
  6. This is the only example of which I am aware of an RBC model that permits a diagrammatic analysis.

Introduction cont.

- The model is stationary to begin with (no trend in technological progress and no population growth), so he is following the first of the two broad strategies I described in my notes cycleres.pdf. "Build a model of the cyclical components of macro time series. Generate predictions, most likely by numerical simulation. The series will be stationary by construction. Use a standard method to remove trends from the data series. Many methods will work. . . . Compare the predictions of the model with the equivalent statistics in the data."

Model specification

- I consider only the version where the representative agent owns capital and rents it to firms.
- There is no population growth and technology is stationary.
- The aggregate production function is given by

$$Y_t = A_t F\left(K_t, N_t\right), \tag{1}$$

with technology A_t following an AR(1) process in logs:

$$\ln\left(A_t\right) = \rho \ln\left(A_{t-1}\right) + \varepsilon_t. \tag{2}$$

Model specification cont.

- The representative household has the following utility function:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t\right) + \nu\left(1 - N_t\right)\right). \tag{3}$$

- The household's budget constraint is given by

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t = w_t N_t + R_t K_t + r_t B_t + \Pi_t. \tag{4}$$

- The household's income is in the form of wages $w_t N_t$, income from renting capital to firms $R_t K_t$, interest payments on bond holdings $r_t B_t$ and dividend payments from profits Π_t.
- The household consumes, invests, and can change its holdings of bonds.

Model specification cont.

- Since there is a representative agent and there is no government (and we assume that firms do not borrow by issuing debt), B_t must actually be zero in equilibrium.
- Everyone in the economy will want either to borrow or lend, and there will be nobody on the other side of the market.
- Leaving this as part of the individual's problem will allow us to evaluate the **riskless** real interest rate r_t.
- There is a (CRS) production function and perfect competition, so payments to factors of production (labour and capital) will be exactly equal to output.
- Therefore, **realized** profits will be zero in every period, and we can assume $\Pi_t = 0$.

Household's problem

- The household's problem can be written as

$$max_{\lambda_t, C_t, N_t, K_{t+1}, B_{t+1}} \mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \left(u\left(c_t\right) + \varphi\left(1 - N_t\right)\right)$$

$$+\lambda_t \left[w_t N_t + R_t K_t + r_t B_t - C_t - K_{t+1} + (1-\delta) K_t - B_{t+1} + B_t \right] \Big\}. \tag{5}$$

Household's problem cont.

- The first order conditions give:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Rightarrow u'(C_t) = \lambda_t; \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Rightarrow \nu'(1 - N_t) = \lambda_t w_t; \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Rightarrow \lambda_t = \beta E_0 \left(\lambda_{t+1} \left(R_{t+1} + (1 - \delta)\right)\right); \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Rightarrow \lambda_t = \beta E_0 \left(\lambda_{t+1} (1 + r_{t+1})\right); \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow \text{Budget constraint holds.} \tag{10}$$

Firm's problem

- If the household is the owner of the firm, it will want to weight future profits by marginal utility
- However, if the firm just rents capital and hires workers each period and has no profits to retain or invest, then its problem is static.
- We can directly write down the first order conditions for the firm's optimal choice of labour and capital:

$$w_t = A_t F_n (K_t, N_t), \tag{11}$$

and

$$R_t = A_t F_k (K_t, N_t). \tag{12}$$

- In the usual way, the real wage equals the marginal product of labour, and the rental rate of capital equals the marginal product of capital.

Equation system

- We want to write down a complete system of equations which must hold in general equilibrium.
- First, substitute out λ_t using the first order condition w.r.t. C_t:

$$\nu'(1 - N_t) = u'(C_t) w_t.$$

- Now substitute out λ_t from the FOCs w.r.t. K_{t+1} and B_{t+1}:

$$u'(C_t) = \beta E_0 \left(u'(C_{t+1}) (R_{t+1} + (1 - \delta)) \right);$$

$$u'(C_t) = \beta E_0 \left(u'(C_{t+1}) (1 + r_{t+1}) \right).$$

- Essentially, we are trying to eliminate some of the variables of the model by substitution.

Equation system cont.

We can then write down the following equation system:

$$u'(C_t) = \beta E_0 \left(u'(C_{t+1}) \left(A_t F_k(K_t, N_t) + (1 - \delta) \right) \right); \tag{13}$$

$$\nu'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t); \tag{14}$$

$$K_{t+1} = A_t F(K_t, N_t) - C_t + (1 - \delta) K_t; \tag{15}$$

$$\ln(A_t) = \rho \ln(A_{t-1}) + \varepsilon_t; \tag{16}$$

$$Y_t = A_t F(K_t, N_t) \tag{17}$$

$$Y_t = C_t + I_t \Rightarrow I_t = Y_t - C_t; \tag{18}$$

$$u'(C_t) = \beta E_t \left(u'(C_{t+1}) (1 + r_{t+1}) \right); \tag{19}$$

$$w_t = A_t F_N(K_t, N_t); \tag{20}$$

$$R_t = A_t F_K(K_t, N_t). \tag{21}$$

Equation system cont.

- This gives nine equations for the following nine unknowns:
  1. the predetermined state variables K_t and A_t;
  2. the non-predetermined or forward-looking dynamic variable C_t;
  3. N_t , r_t, Y_t, I_t, w_t, R_t.
- We know which variables are **state** and **dynamic** variables because they appear in the equation system with different dates: t and $(t+1)$. They define an explicit or implicit difference equation.
- Six static variables appear with just one date.
- From equations (17), (20), and (21), Y_t, w_t and R_t are determined **recursively**.
- If we have solve for the other variables of the system, we can then compute their values.
- Equation (18) shows that I_t is semi-recursive. If we solve for the other variables of the system we can compute Y_t and then we can compute I_t.

Equation system cont.

- Finally, equation (19) allows us to determine the real interest rate in the economy.
- From (13) and (19) the expected rates of return on investment and on increasing bond holdings have to be equal.
- But we know from our reasoning that B_t does not appear in the equation system because it must be equal to zero in equilibrium.
- If we get rid of the variables which are determined recursively and r_t, we have a four-equation system to determine K_t, C_t, A_t — and N_t, which is determined by the static FOC (14).
- When using a phase diagram, we will consider A_t to be exogenous.

Functional forms

- Sims then imposes functional forms on the utility function and the production function:

$$u\left(C_t\right) = \ln\left(C_t\right);$$

$$\nu\left(1 - N_t\right) = \theta \ln\left(1 - N_t\right);$$

$$Y_t = A_t {K_t}^{\alpha} {N_t}^{(1-\alpha)}.$$

- We are now ready to start talking about the steady state.

Steady state

- Impose constant values of the variables in the model. Use asterisks to denote these values.
- In particular, $A_t = A_{t-1} = A^*$. From (16) it's clear that the long-run value of the log of A is zero, so $A^* = 1$.
- Also impose $K_{t+1} = K_t = K^*$ and $C_{t+1} = C_t = C^*$.
- We can first find the long-run value of capital to employment ratio, which we will then use to solve for a bunch of other stuff. From (13):

$$1 = \beta \left(\alpha \left(\frac{K^*}{N^*} \right)^{(\alpha-1)} + (1-\delta) \right);$$

$$\Rightarrow \frac{1}{\beta} - (1-\delta) = \alpha \left(\frac{K^*}{N^*} \right)^{(\alpha-1)};$$

$$\Rightarrow \left(\frac{K^*}{N^*} \right) = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right).$$

Steady state cont.

- The strategy will be to try to simplify as many of the other conditions as possible by expressing them in terms of the capital to employment ratio.
- We immediately have

$$w^* = (1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha},$$

and

$$R^* = \alpha\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}.$$

- We also have from the FOCs for B_{t+1} and K_{t+1} that

$$1 = \beta\left(R^* + (1-\delta)\right) = \beta(1+r^*) \Rightarrow R* = r* + \delta.$$

Steady state cont.

- We saw the same relationship between the interest rate and the rental rate of capital Romer's RBC model. The return to capital must cover the costs of financing investment (the real interest rate) and the costs of depreciation.
  Equations (15) and (17) together imply

$$I^* = \delta K^*.$$

- We will need a solution for N^* in order to be able to calculate I^* by itself.

Steady state cont.

- We do this as follows. From the production function and the accounting identity (17) we have

$$K^{*\alpha} N^{*(1-\alpha)} = C^* + I^*$$

$$\Rightarrow \left(\frac{K^*}{N^*}\right)^{\alpha} N^* = C^* + \delta K^*$$

$$\Rightarrow \frac{C^*}{N^*} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta \left(\frac{K^*}{N^*}\right).$$

- Use (14) to get a second expression for C^*/N^*, then equate the two expressions:

$$\frac{\theta}{1-N^*} = \frac{1}{C^*}(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha}$$

$$\Rightarrow \frac{C^*}{N^*} = \frac{1-N^*}{N^*}\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha}.$$

Steady state cont.

- Equating the two expressions for C^*/N^* gives

$$\frac{1-N^*}{N^*}\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right).$$

- Multiplying through by N^* gives

$$\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} - N^*\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} = N^*\left(\frac{K^*}{N^*}\right)^{\alpha} - N^*\delta\left(\frac{K^*}{N^*}\right).$$

- Solving for N^* gives

$$N^* = \frac{\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha}}{\frac{\theta+1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right)}. \tag{22}$$

Steady state cont.

- This is pretty messy, but we've found employment as a function of the capital to employment.
- The rest of the steady state solutions are now pretty easy. For I^* we have:

$$I^* = \delta K^* = \delta \left(\frac{K^*}{N^*} \right) N^*. \tag{23}$$

- For Y^* we have:

$$Y^* = K^{*\alpha} N^{(1-\alpha)} = \left(\frac{K^*}{N^*} \right)^{\alpha} N^*. \tag{24}$$

- For C^* we have:

$$C^* = Y^* - I^* = \left(\frac{K^*}{N^*} \right)^{\alpha} N^* - \delta \left(\frac{K^*}{N^*} \right) N^*.$$

Calibration

- The underlying parameters of the model are: β, α, θ, and δ.
- β: the long-run level of the real interest rate (see why we needed it?), we have:

$$1 = \beta\left(1 + r^*\right) \Rightarrow r^* = \frac{1}{\beta} - 1.$$

- With any $\beta < 1$, there is a positive real (riskless) interest rate. If we think the average annual real interest rate is about two percent, this gives 0.005 for quarterly data and $\beta = 0.995$.
- α: it's the share of capital in GDP. This is around 1/3 in most countries.

Calibration cont.

- δ: we could choose δ based on national accounts, which would imply a value for the share of investment in output. Or we can proceed in reverse, using the share of investment in output to pin down δ. Use

$$\frac{I^*}{N^*} = \delta \frac{K^*}{N^*} \text{ and } \frac{Y^*}{N^*} = \left(\frac{K^*}{N^*}\right)^{\alpha}.$$

$$\Rightarrow \frac{I^*}{Y^*} = \delta \left(\frac{K^*}{N^*}\right)^{(1-\alpha)}.$$

$$\Rightarrow \frac{I^*}{Y^*} = \frac{\delta\alpha}{\frac{1}{\beta} - (1-\delta)}.$$

- I^*/Y^* is about 20%. Sims rounds to 22.5%. Using the values of α and β we get $\delta \approx 0.02$. This gives $K^*/N^* \approx 35$.

Calibration cont.

- This leaves θ. From the capital accumulation equation in the steady state,

$$\frac{C^*}{N^*} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}.\right)$$

- The FOC for labour supply gives a different expression for C^*/N^*:

$$\frac{C^*}{N^*} = \frac{1-N^*}{N^*}\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha}.$$

- Equating the two gives

$$\frac{1}{\theta}\frac{1-N^*}{N^*}(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right),$$

$$\Rightarrow \theta = \frac{\frac{1-N^*}{N^*}(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha}}{\left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right)}.$$

- If we set $N^* = 1/3$ we get $\theta \approx 1.75$.

Stochastic process for A_t

- Sims gets TFP from the data. In the model it is stationary, so Sims uses deviations from and estimated linear trend.
- This is an example, as we have already noted, of using different assumptions about nonstationarity in the model and the data.
- He estimates

$$\ln\left(\hat{A}_t\right) = \phi_0 + \phi_1 t + u_t.$$

  He gets $\hat{\phi}_0 = 0.145$ and $\hat{\phi}_1 = 0.003$.
- He then takes the residual of the equation and estimates

$$\hat{u}_t = \rho \hat{u}_{t-1} + e_t.$$

- He gets a $\hat{\rho} = 0.974$ with a standard error of 0.009.

Phase diagram

- Sims solves for the "isoclines" of the model, curves along which either C_t is not changing of K_t is not changing.
- We must take A_t to be fixed or exogenous to get a two-dimensinal system.
- The procedure is: (1) linearize the model; (2) calculate the isoclines; and (3) find the directions of motion of C and K on either side of the isoclines.
- The two main dynamic equations of the model come from the law of motion for the capital stock; and the consumption Euler equation.
- It is necessary to eliminate all endogenous variables other than C_t and K_t by substitution. Sims demonstrates this in detail. The main variable to substitute out is N_t.

Linearized equation for N_t

- Write the condition for labour supply in log form:

$$\ln(\theta) - \ln(1 - N_t) = -\ln(C_t) + \ln(1 - \alpha) + \ln(A_t) + \alpha \ln(K_t) - \alpha \ln(N_t).$$

- Rewrite in terms of proportional deviations from the long run:

$$\frac{N_t - N^*}{1 - N^*} = -\frac{C_t - C^*}{C^*} + \frac{A_t - A^*}{A^*} + \alpha \frac{K_t - K^*}{K^*} - \alpha \frac{N_t - N^*}{N^*}.$$

- Use a more compact notation, defining for any X_t, $\tilde{X}_t \equiv \frac{X_t - X^*}{X^*}$:

$$\frac{N^*}{1 - N^*} \frac{N_t - N^*}{N^*} \equiv \frac{N^*}{1 - N^*} \tilde{N}_t = -\tilde{C}_t + \tilde{A}_t + \alpha \tilde{K}_t - \alpha \tilde{N}_t.$$

Linearized equation for N_t cont.

- Define $N^* / (1 - N^*) \equiv \gamma$ to make the notation more compact. We have:

$$\tilde{N}_t = -\frac{1}{\gamma + \alpha} \tilde{C}_t + \frac{1}{\gamma + \alpha} \tilde{A}_t + \frac{\alpha}{\gamma + \alpha} \tilde{K'}_t, \tag{25}$$

- This will allow us to substitute out $\tilde{N}_t$ in the Euler equation and the law of motion for capital.

Euler equation

- Write in logs:

$$-\ln(C_t) = \ln(\beta) - \ln(C_{t+1}) + \ln\left(\alpha A_{t+1} K_{t+1}^{(\alpha-1)} N_{t+1}^{(1-\alpha)} + (1-\delta)\right).$$

- Linearize, ignore zero-order terms and make use of $\alpha K^{*(\alpha-1)} N^{*(1-\alpha)} + (1-\delta) = 1/\beta$. I leave out a couple of steps here:

$$-\tilde{C}_t = -\tilde{C}_{t+1} + \beta\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}\left(\tilde{A}_{t+1} + (\alpha-1)\tilde{K}_{t+1} + (1-\alpha)\tilde{N}_{t+1}\right).$$

- Forward (24), use to eliminate $\tilde{N}_{t+1}$. After tedious algebra:

$$-\tilde{C}_t = -\left(1 + \beta(1-\alpha)\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}\right)\tilde{C}_{t+1}$$

$$+\left(\beta\alpha\frac{K^*}{N^*}\right)^{(\alpha-1)}\left(\frac{1+\gamma}{\gamma\alpha}\right)\tilde{A}_{t+1} - \left(\beta\alpha(1-\alpha)\frac{K^*}{N^*}\right)^{(\alpha-1)}\left(\frac{\gamma}{\gamma\alpha}\right)\tilde{K}_{t+1}. \quad (26)$$

Euler equation cont.

- This is messy. Set $\tilde{C}_t = \tilde{C}_{t+1}$ to calculate the "isocline," along which $\tilde{C}$ doesn't change. Once again, the algebra is messy, but yields

$$\tilde{C}_{t+1} = \left(\frac{1+\gamma}{1-\alpha}\right)\tilde{A}_{t+1} - \gamma\tilde{K}_{t+1}.$$

- everything is in discrete time, but for very small time intervals we can replace this with an equation with variables dated t:

$$\tilde{C}_t = \left(\frac{1+\gamma}{1-\alpha}\right)\tilde{A}_t - \gamma\tilde{K}_t. \tag{27}$$

- This gives a negatively-sloped relationship between $\tilde{C}$ and $\tilde{K}$. Also, if there is an exogenous increase in $\tilde{A}$, the relationship shifts up. This gives the downward-sloping curve on page 14 of Sims.

Euler equation cont.

- Stare at equation (25) to figure out what happens on either side of the isocline.
- Sims: "Quite intuitively, if $\tilde{K}_t$ is "too big" relative to what it would be when $\tilde{C}_{t+1} - \tilde{C}_t = 0$ — i.e. we are to the right of the $\tilde{C}_{t+1} - \tilde{C}_t = 0$ isocline — then consumption will be expected to decline over time."
- This gives the downward-pointing arrows to the right of the isocline, and the upward-pointing arrows to the left.

Law of motion for capital

- Take logs:

$$\ln\left(K_{t+1}\right) = \ln\left(A_t F\left(K_t, N_t\right) - C_t + (1-\delta)K_t\right).$$

- Linearize with a Taylor expansion, ignore the zeroth order terms, use the compact notation, and substitute out $\tilde{N}_t$ as before to get

$$\tilde{K}_{t+1} = \left(\frac{1}{\beta} + \frac{1-\gamma}{\gamma+\alpha}\alpha\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}\right)\tilde{K}_t + \left(\frac{1+\gamma}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}\right)\tilde{A}_t.$$

$$- \left(\frac{C^*}{K^*}\frac{1-\alpha}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{(\alpha-1)}\right)\tilde{C}_t \tag{28}$$

Law of motion for capital cont.

- When $\tilde{K}_{t+1} = \tilde{K}_t$ this gives a relationship between $\tilde{K}$ and $\tilde{C}$ along which the capital stock is constant:

$$\tilde{C}_t = \left(\frac{C^*}{K^*} \frac{1-\alpha}{\gamma+\alpha} \left(\frac{K^*}{N^*} \right)^{(\alpha-1)} \right)^{-1} \left[\left(\frac{1}{\beta} - 1 + \frac{1-\gamma}{\gamma+\alpha} \alpha \left(\frac{K^*}{N^*} \right)^{(\alpha-1)} \right) \tilde{K}_t + \left(\frac{1+\gamma}{\gamma+\alpha} \left(\frac{K^*}{N^*} \right)^{(\alpha-1)} \right) \tilde{A}_t \right]. \tag{29}$$

- This is positively-sloped. It is possible to show that the slope is less than one. The argument is somewhat convoluted: see Sims starting on page 13.

Law of motion for capital cont.

- We have the upward-sloping isocline on page 14.
- Above consumption is increasing, so savings and investment must be less and the capital stock must be decreasing. Below the capital stock must be increasing.
- This gives the arrows on the phase diagram.
- To prove saddlepoint stability we would have to write this in matrix form with $\tilde{C}_{t+1} - \tilde{C}_t$ and $\tilde{K}_{t+1} - \tilde{K}_t$ as a 2×1 vector on the left hand side, and show that the determinant of the matrix on the right hand side is negative (one negative characteristic root and one positive root).
- Just by staring at the arrows we can deduce the nature of the convergent arm of the saddle, which is the dotted line on the phase diagram.

Shocks

- Sims looks at the effects of a permanent increase in A_t.
- This violates the assumption that technology is stationary and hence there is no growth.
- The phase diagram is on page 19. The long-run equilibrium level of the capital stock and consumption increase.
- Both isoclines shift up and to the right. The convergent arm of the saddle shifts up.
- Consumption jumps up immediately to place the economy on the convergent arm, and then consumption and capital slowly increase along the path towards the new long-run equilibrium.
- The time paths of consumption and the capital stock are illustrated on page 19. Sims goes on to look at what this implies for the dynamics of the real wage and employment.

Conclusion

- We analyzed Sims' version of the baseline RBC model.
- We reviewed how to solve the model analytically for the steady state.
- We discussed the relationship between the steady state solution and the parameter values of the model.
- We briefly discussed how to linearize the model and how to analyze its dynamics using phase diagrams.
- I think the Sim's article is useful as a teaching tool for the these reasons.

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