Solow Growth Model

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Introduction

- A copy of these slides is available at
  `https://steveambler.uqam.ca/511/solows.pdf`
- All documents for the course are available in the following subdirectory
  `https://steveambler.uqam.ca/511/`
- I will also be uploading this file to CANVAS

Key works

- Kaldor's stylized facts
- Constant returns to scale (CRS)
- Diminishing marginal returns
- Cobb-Douglas production function
- Intensive form of the production function
- Balanced growth path
- Convergence
- Growth accounting
- Solow residuals
- Endogenous growth
- Research and development
- Non-rivalry, exclusivity
- Learning by doing
- Human capital
- Social infrastructure
- Externalities
- Rent seeking

The importance of economic growth

- Impressive growth in income levels since the Industrial Revolution
- Drastic reduction in absolute poverty levels over the last 30 years
- `humanprogress.org`
- Lucas (1987): increasing average growth rates by one percentage point has a much greater positive impact on welfare than the complete elimination of the business cycle
- Reading Lucas (1987) is well worth it. We will return to the question in Section 7 of the course

Kaldor's stylized facts of growth

- Kaldor (1961). See also Wikipedia, "Kaldor's facts"
- Mostly advanced economies, but also emerging economies
  1. The shares of national income received by labour and capital are roughly constant over long periods of time
  2. The rate of growth of the capital stock per worker is roughly constant over long periods of time
  3. The rate of growth of output per worker is roughly constant over long periods of time
  4. The capital/output ratio is roughly constant over long periods of time
  5. The rate of return on investment is roughly constant over long periods of time
  6. There are appreciable variations (2 to 5 percent) in the rate of growth of labor productivity and of total output among countries.
- For "roughly constant" read "mean reverting"

More facts

1. Y/L 10 to 30 times greater than 150 years ago
2. Y/L 50 to 300 times greater than 250 years ago
3. Growth slowdown since 1970 in many countries
4. Huge differences across countries of Y/L
5. "Growth miracles": Japan 1945-1990, South Korea, Taïwan, Singapore, Hong Kong 1960-1990
6. "Growth disasters": Argentina, Venezuela, Sub-Saharan Africa until recently

Basic Solow model

- Compatible with many of Kaldor's facts
- Main conclusion: physical capital accumulation **cannot** explain the growth in per capita GDP or gaps in per capita GDP across countries
- Aggregate production function:

$$Y(t) = F\left(K(t), A(t)L(t)\right)$$

- We drop time subscripts if there is no ambiguity

Basic Solow model 2

- Constant returns to scale (CRS):

$$F(cK\ ,\ cAL) = c\,F(K\ ,\ AL) \quad \forall c \geq 0$$

- **Intensive form** of the production function:

$$c = 1/(AL) \Rightarrow Y/(AL) = F\left(\frac{K}{AL}, 1\right)$$

- In more compact notation:

$$y = f(k),\ y \equiv Y/(AL),\ k \equiv K/(AL)$$

- y is a concave function of k
- With a Cobb-Douglas production function:

$$Y = K^{\alpha}\,(A\,L)^{1-\alpha} \Rightarrow y = k^{\alpha}$$

Hypotheses

- One instance where continuous time is easier.
  1. $\dot{L} = n\,L$
  2. $\dot{A} = g\,A$
  3. $\dot{K} = I - \delta\,K = s\,Y - \delta\,K$
- Here, s est the savings rate, δ is the depreciation rate, and for any variable X:

$$\dot{X} \equiv \frac{\partial X}{\partial t}$$

- Savings rate s is constant and exogenous

Evolution of k

$$
\begin{aligned}
\dot{k} &= \dot{\left(\frac{K}{LA}\right)} \\
&= \frac{\dot{K}}{LA} - \frac{K}{(AL)^2}\left(A\dot{L} + \dot{A}L\right) \\
&= \frac{\dot{K}}{LA} - \frac{K}{LA}\frac{\dot{L}}{L} - \frac{K}{LA}\frac{\dot{A}}{A} \\
&= \frac{s\,Y - \delta\,K}{LA} - k\,n - k\,g \\
&= s\frac{Y}{AL} - \delta\,k - n\,k - g\,k \\
\Rightarrow \dot{k} &= s\,f(k) - (n + g + \delta)\,k.
\end{aligned}
$$

- In the first line, the dot over $\left(\frac{K}{LA}\right)$ applies to the whole parenthesis, so it's the rate of change of the ratio

Discussion

- First term on RHS is equal to investment (in a closed economy)
- Second term has the interpretation of the investment necessary to maintain a constant capital stock per unit of effective labour:
  1. replace depreciating capital;
  2. supply additional capital to new workers;
  3. supply additional capital to workers who become more productive
- See Figure 1.2 in Romer (2018)
- One dynamic variable (k). We can represent the dynamics with a simple "phase diagram"
- See Figure 1.3 in Romer (2018)

Balanced growth path

- In long run (with no changes in exogenous parameters):

$$\dot{k} = 0 \Rightarrow k = k^*.$$

  Use an asterisk to indicate capital per unit of effective labour along a balanced growth path
- We get:

$$\begin{aligned}
K &\equiv A\,L\,k \\
\Rightarrow \dot{K} &= \dot{A}\,L\,k + A\,\dot{L}k + A\,L\,\dot{k} \\
&= \dot{A}\,L\,k + A\,\dot{L}k \\
\Rightarrow \frac{\dot{K}}{K} &= \frac{\dot{A}}{A}\frac{AL}{K}k + \frac{\dot{L}}{L}\frac{AL}{K}k = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} \\
\Rightarrow \frac{\dot{K}}{K} &= n + g \\
\Rightarrow \frac{\dot{Y}}{Y} &= n + g
\end{aligned}$$

Consequences

- Y/L, K/L grow at rate g (exercise)
- The growth rate of per capita output depends only on g
- Comparing to Kaldor's stylized facts:
  1. K/Y is constant;
  2. K/L grows over time;
  3. Y/L grows over time;
  4. In the Cobb-Douglas case

$$\frac{\partial Y}{\partial K} = \frac{\partial K^{\alpha}(AL)^{(1-\alpha)}}{\partial K} = (AL)^{(1-\alpha)} K^{(\alpha-1)} = \left(\frac{K}{AL}\right)^{(\alpha-1)} = k^{(\alpha-1)}.$$

Since k is stable in the long run, the marginal productivity of capital (and hence the real interest rate) is stable in the long run'

- We speak of **balanced growth** because aggregates (Y, C, K, I) grow at the same rate ($n+g$) and aggregates per unit of effective labour (y, c, k, i) grow at the same rate (g).

Impact of a change in the savings rate

- See Figure 1.4 in Romer (2018)
- See Figure 1.5 in Romer (2018)

Impact on consumption

- See figure 1.6 in Romer
- There is a savings rate which **maximizes** c (C/AL) in the steady state (**golden rule level**)
- If we start with a level of k **below** that which maximizes c, some consumption must be sacrificed to reach the golden rule level
- Because of this, in a model with intertemporal maximization, the golden rule is not necessarily optimal
- Because the sacrifices are made **now** and the benefits occur **later**, it's only if household discount rates are zero that it becomes optimal to aim for the golden rule
- We will instead speak of a **modified golden rule**

Quantitative implications: impact of savings on long-term production

- The long-term capital stock depends on the exogenous parameters of the model:

$$k^* = k^*(s\,,\,n\,,\,g\,,\,\delta)$$

- The impact of a change in s on y in the long run is given by:

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*(s\,,\,n\,,\,g\,,\,\delta)}{\partial s}$$

- We have to evaluate the partial derivative $\partial k^* / \partial s$

Impact of savings on long-run production 2

- k^* satisfies:

$$\begin{aligned} s\,f(k^*(s\,,\,n\,,\,g\,,\,\delta)) &= (n+g+\delta)k^*(s\,,\,n\,,\,g\,,\,\delta) \\ \Rightarrow s\,f'(k^*(s\,,\,n\,,\,g\,,\,\delta))\frac{\partial k^*}{\partial s} + f(k^*) &= (n+g+\delta)\frac{\partial k^*}{\partial s} \\ \Rightarrow \frac{\partial k^*}{\partial s} &= \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)} \end{aligned}$$

- Substituting in the equation for $\partial k^*/\partial s$ in the last equation of the previous slide, we get:

$$\frac{\partial y^*}{\partial s} = f'(k^*)\frac{f(k^*)}{(n+g+\delta) - sf'(k^*)}$$

Impact of savings on long-run production 3

- Convert to an **elasticity**:

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \frac{s}{f(k^*)}f'(k^*)\frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$$
$$= \frac{f'(k^*)}{f(k^*)}\frac{sf(k^*)}{(n+g+\delta)-sf'(k^*)}$$

- Now use the fact that $s\,f(k^*) = (n+g+\delta)k^*$.

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \frac{f'(k^*)\,(n+g+\delta)\,k^*}{f(k^*)\left[(n+g+\delta)-(n+g+\delta)k^*f'(k^*)/f(k^*)\right]}$$
$$= \frac{k^*f'(k^*)/f(k^*)}{1-k^*f'(k^*)/f(k^*)}$$
$$\equiv \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$$

Impact of savings on long-run production 4

- In perfect competition, capital is paid its marginal product
- So, α_K is the share of capital in national revenue (or in GDP)
- In the data, $\alpha_K \approx 1/3$ (calibration!)
- This gives:

$$\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} \approx \frac{1}{2}$$

- Large variations in s have moderate or weak effects on output per unit of effective labour
- Suppose $s = 0.20$ initially and it increases by 10% (from 0.20 to 0.22) $\frac{\partial s}{s} = 0.10$, so therefore $\partial y/y = 0.05$, an increase in GDP of 5%

Speed of convergence

- A first-order Taylor expansion gives:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} \right] (k - k^*)$$

We write only first-order terms (the zero-order term is zero)

- We get:

$$\begin{aligned}
\left[\frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} \right] &= sf'(k^*) - (n + g + \delta) \\
&= \frac{(n + g + \delta)\, k^*}{f(k^*)} f'(k^*) - (n + g + \delta) \\
&= \left(\frac{k^*}{f(k^*)} f'(k^*) - 1 \right) (n + g + \delta) \\
&= [\alpha_K - 1]\, (n + g + \delta) \\
\Rightarrow \dot{k} &\approx [\alpha_K - 1]\, (n + g + \delta)\, (k - k^*)
\end{aligned}$$

Speed of convergence 2

- To find the **growth rate** of k, divide by k to get

$$\frac{\dot{k}}{k} \approx [\alpha_K - 1] \, (n + g + \delta) \, (k - k^*) \, ,$$

which is proportional to $(k^* - k)$
- Capital per unit of effective labour converges towards k^* at a rate proportional to the gap between k et k^*.
- If $(k < k^*)$, the capital stock is increasing and we can write

$$\dot{k} = [1 - \alpha_K] \, (n + g + \delta) \, (k^* - k)$$

- Let's attribute plausible numerical values (calibration!): n 1-2%, g 1-2%, δ 3-4%. We get

$$(n + g + \delta) \approx .06 \Rightarrow (1 - \alpha_K) \, (n + g + \delta) \approx .04.$$

- So, about 4% of the gap between k and k^* is eliminated each year
- So it takes about 18 years to eliminate half of the gap (rule of 70). This is a **very slow** speed of convergence

Problems with the model

- Can we explain observed gaps in GDP per capita?
- Say Canadian GDP per capita is about 10 times that of India. We get

$$\frac{y_1}{y_2} = 10 \Rightarrow \frac{{k_1}^{\alpha_k}}{{k_2}^{\alpha_k}} = 10$$

$$\Rightarrow \frac{k_1}{k_2} = 10^{1/\alpha_K} \approx 10^3 = 1000$$

- However, the Canadian capital stock (per capita) is about 10 times that of India
- Conclusion: differences in per capita GDPs must be explained by differences in the levels of the A's (exogenous differences in productivity)
- This isn't really an **explanation** in scientific terms
- We end up explaining growth by the main **exogenous** factor in the model
- This goes a long way to explaining the development of **new growth theory** or **endogenous growth theory** in the 1990s

Growth accounting

- See Solow (1957), Abramovitz (1956)
- We have

$$\begin{aligned} \dot{Y} &= \frac{\partial Y}{\partial K}\dot{K} + \frac{\partial Y}{\partial L}\dot{L} + \frac{\partial Y}{\partial A}\dot{A} \\ \Rightarrow \frac{\dot{Y}}{Y} &= \frac{K}{Y}\frac{\partial Y}{\partial K}\frac{\dot{K}}{K} + \frac{L}{Y}\frac{\partial Y}{\partial L}\frac{\dot{L}}{L} + \frac{A}{Y}\frac{\partial Y}{\partial A}\frac{\dot{A}}{A} \\ &\equiv \alpha_K \frac{\dot{K}}{K} + \alpha_L \frac{\dot{L}}{L} + R \end{aligned}$$

- With perfect competition, $\alpha_K + \alpha_L = 1$, so

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \alpha_K \left[\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right] + R.$$

- R is the **Solow residual** (often called "z")
- The Solow residual is still used by statisticians and economists
- It is the basis of the real business cycle (RBC) model

More on convergence

- Income gaps disappear for 3 reasons:
  1. Convergence to balanced growth;
  2. Capital flows to poorer countries;
  3. Diffusion of knowledge leading to convergence of A's.
- Evidence from Baumol (1986). Data from 16 industrialized countries: Japan, Sweden, Finland, Norway, Germany, Canada, Austria, France, Italy, Netherlands, US, Denmark, Switzerland, Belgium, UK, Australia
- He assumes implicitly that the **only** recent for an initial difference in per capita GDP in 1870 is due to different starting points with respect to the long run equilibrium, which is the same for everyone

More on convergence 2

- The equation he estimates is

$$\ln\left[\left(\frac{Y}{L}\right)_{i,1979}\right] - \ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right] = a - b\ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right] + u_t$$

- He gets the following estimates for a and b:

$$\ln\left[\left(\frac{Y}{L}\right)_{i,1979}\right] - \ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right] = 8.457 - 0.995\ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right],$$

  $R^2 = 0.87$; the standard deviation of the regression coefficient ($\hat{b} = -0.995$) is 0.094, so it is **highly** significant

- So convergence is **significant**

More on convergence 3

- Criticism by DeLong (1988):
  1. Selection bias: countries which were poor in 1870 are in the sample **only** if they have had an elevated growth rate since
  2. Measurement error: if we overestimate the **level** of income in 1870 we will **underestimate** the growth rate since then. Measured growth lower in countries with high initial income **even if** there is no real relation
- We need a statistical model of the error term. DeLong assumes:

$$\begin{aligned}
\ln\left[\left(\frac{Y}{L}\right)_{i,1979}\right] - \ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right]^* &= a - b\ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right]^* + \varepsilon_i, \\
\ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right] &= \ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right]^* + u_i, \\
\mathrm{Corr}(\varepsilon_i\,,\, u_i) &= 0.
\end{aligned}$$

More on convergence 4

- Here,

$$\ln\left[\left(\frac{Y}{L}\right)_{i,1870}\right]^*$$

is the **true** value of income per capita and u_i is the measurement error

- DeLong adds 7 countries to create a more representative sample (Argentina, Chile, East Germany, Ireland, New Zealand, Portugal, Spain) and removes Japan
- We can't really identify the size of the u_i error using data. We have to make an **assumption**
- If $\sigma_u = 0.15$ we get $\hat{b} = 0$ (**no convergence**). If $\sigma_u = 0.20$, we get $\hat{b} = 1$ (**divergence**)

More on convergence 6

- Bottom line: for reasonable values, there is **no convergence**
- See Figure 1.9 for per capita GDP in 1970 and growth from 1970 to 2014 (fifth edition) or per capita GDP in 1970 and growth from 1970 to 2003 (fourth edition). There is no apparent relationship discernible from the scatter diagram
- Baumol (and DeLong) have a **univariate**) explanation for convergence (initial per capita GDP). There are omitted variables which could bias the results: economic policy variables, education, etc.
- Mankiw, Romer and Weil (1992) add **human capital** and get much more encouraging results. We will return to this question in the next chapter

The environment, natural resources and economic growth

- We skip this section because there's not enough time

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