Inflation Targeting, Price-Level Targeting, the Zero Lower Bound, and Indeterminacy*

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Abstract

We compare inflation targeting and price-level targeting in the canonical New Keynesian model, with particular attention to multiple steady-states, indeterminacy, and global stability. Under price-level targeting we show the following: 1) the well-known problem of multiple steady-state equilibria under inflation targeting is absent; 2) the model's dynamics close to the steady state are determinate for a much wider range of parameter values; 3) the model is globally saddlepoint stable. These results provide additional arguments in favour of price-level targeting as a monetary policy framework.

JEL classification: E31, E32, E52

Keywords: Monetary policy; Price-level targeting; Inflation targeting; Indeterminacy

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1 Introduction

An extensive literature comparing the performance of inflation targeting (henceforth IT) and price-level path targeting (henceforth PT)[1] highlights the superiority of the latter as a stabilization tool. The volatility of both inflation and output are lower under PT than under IT, as long as expectations are rational and the central bank's announced policies are credible.[2] PT works by guiding future expectations of future policy and inflation. In response to a positive inflation shock, future inflation is expected to be below target in order for the price level to revert to its targeted path. Firms increase their prices less, so inflation is less volatile. Because of this, the central bank needs to reduce the output gap by less to bring inflation down, and output is less volatile.

Many of the studies comparing the performance of IT and PT have used New Keynesian models solved by approximating agents' first order conditions in the neighbourhood of a zero-inflation deterministic steady state. Some studies take into account the zero lower bound on the central bank's policy rate while still approximating first order conditions around the zero-inflation steady steady state.[3]

[1] Ambler (2009) surveys the literature on PT that discusses its potential advantages and disadvantages compared to IT, and Ambler (2014) discusses why PT has not yet been tried out by any central bank despite these advantages.

[2] The optimal (Ramsey) interest rate rule in the canonical Keynesian model implies a stationary price level as shown by Clarida, Galí and Gertler (1999) and Woodford (1999). This suggests that simple PT rules (which give a stationary price level) may do better than simple IT rules. Vestin (2006) shows in a simple model that a discretionary central bank that minimizes a loss function defined in terms of the price level can attain the same level as a central bank that maximizes the true social welfare function under commitment.

[3] Adam and Billi (2006) linearize the equations of the model and then use projection methods to account for the zero bound on the central bank's policy rate. Amano and Ambler (2014) use

This ignores another potential disadvantage of IT, the existence of multiple steady states. Benhabib, Schmitt-Grohé and Uribe (2001 and 2001b, henceforth BSU) showed that the zero lower bound on the central bank's policy rate implies that under IT there must be two deterministic steady-state equilibria.[4] The literature comparing IT and PT has focused for the most part on the equilibrium where the central bank achieves its target for the inflation rate while ignoring the second "liquidity-trap" equilibrium where the nominal interest rate is stuck at or near the zero bound and the inflation rate is negative. In the New Keynesian model this implies a large negative output gap.

Mendes (2011) recently conjectured that history-dependent policy rules could eliminate the multiplicity of steady-state equilibria (he focused on stochastic steady states) and demonstrated that this is the case for a simple rule where the central bank's desired policy rate depends negatively on the time spent at the zero bound. The PT regime is an example of history dependence since past inflation surprises are corrected or offset by the central bank. This suggests that PT may offer an additional advantage over IT by eliminating low-inflation equilibria.

This paper compares IT and PT in the canonical New Keynesian model with particular attention to the existence of multiple steady states, regions of indeterminacy in parameter space, and global stability. We demonstrate the following results.

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higher-order approximations and use a smooth approximation to the kink in the central bank's reaction function at the zero bound.

[4] Schmitt-Grohé and Uribe (2009) show that self-fulfilling liquidity traps cannot be ruled out under IT even if the central bank's policy rate is not constrained to be positive.

1. There is only one deterministic steady-state equilibrium under PT.

2. Under PT, there exists a second "quasi steady state" in the deterministic case in which the gap between the price level and its optimal path grows without bound over time. This quasi steady state corresponds to the low-inflation deterministic steady state in an IT regime.

3. Under PT, there can only be one stochastic steady state equilibrium. If the policy rate is at the lower bound, then insofar as agents expect that the interest rate will eventually leave the lower bound in response to a positive shock to inflation, the central bank's commitment to moving the price level back to its target path entails that the unconditional expectation of inflation is the central bank's target inflation rate. This eliminates the possibility of the second "quasi steady state."

4. The model's dynamics are determinate near the steady-state equilibrium for a much wider range of parameter values under PT than under IT. In particular, determinacy is less sensitive to the parameter values of the central bank's interest rate reaction function, including the strength of its reaction to deviations of the price level from the target path.

5. Using backward integration (following Brunner and Strulik, 2002) to solve the model, we verify numerically that the model economy is globally saddlepoint stable under PT.

Taken together, these results provide additional strong arguments to favour PT over IT as a monetary policy framework. In particular, PT eliminates the possibility of bad steady-state equilibria and reduces the likelihood of indeterminate dynamics.

We outline our model in section 2. Section 3 shows that our model, like that of BSU, has two deterministic steady states under IT. Section 4 shows that under PT the model has a unique deterministic steady state, but also has a "quasi steady state" with a price level gap that increases over time. Section 5 discusses the model's stochastic steady states under IT (summarizing the results of Mendes, 2011) and under PT. Section 6 analyzes the determinacy properties of the model near the high-inflation steady state as a function of parameter values. Section 7 presents a constructive proof that the model is globally determinate under PT. Section 8 concludes. Details of proofs are relegated to a technical appendix.

2 Theoretical Framework

We consider the canonical New Keynesian macroeconomic model given by the following set of three equations.[5]

[5] We follow much of the literature and Adam and Billi (2006) and Mendes (2011) in particular in using linear equations except for the central bank's interest rate reaction function. The crucial feature of the model for our results is a policy function for the interest rate (a Taylor rule or a modified Taylor rule) that is convex because of the zero lower bound constraint, and more convex than the Fisher relationship between the inflation rate and the nominal interest rate. Using nonlinear versions of the Phillips curve and IS equation would complicate the story without changing the results. See Galí (2008) for a detailed derivation of the equations of the standard New Keynesian model.

The New Keynesian Phillips curve is given by

$$\pi_t = (1-\beta)\pi^* + \beta \mathrm{E}_t \pi_{t+1} + \varphi y_t, \tag{1}$$

where π_t is inflation, π^* is trend or target inflation, y_t is the output gap, and E_t is the mathematical expectations operator conditional on information available at time t.[6] We assume here that $\pi^* > 0$, so that the central bank targets a positive inflation rate in the long run.[7]

The New Keynesian IS equation given by

$$y_t = \mathrm{E}_t y_{t+1} - \frac{1}{\gamma}\left(i_t - \mathrm{E}_t \pi_{t+1} - r_t\right) + v_t, \tag{2}$$

where r_t is the natural real interest rate, i_t is the short-term nominal interest rate, set directly by the central bank, and v_t is a demand shock.

Under IT, the model is completed by the following Taylor rule:

$$i_t^d = r_t + \pi^* + \rho_\pi\left(\pi_t - \pi^*\right) + \rho_y y_t, \tag{3}$$

where i_t^d is the desired nominal rate of interest, and where $\rho_\pi > 1$ so that the

[6] We could add a cost-push shock to this equation, but this would in no way change the results of our paper.

[7] Mendes (2011) considers negative values of π^*, and shows that the Friedman rule is not feasible in the presence of stochastic shocks to the real rate of interest.

Taylor principle is satisfied.[8] The actual nominal rate of interest is given by

$$i_t = \max\left(0, i_t^d\right), \tag{4}$$

so that the nominal interest rate is subject to a zero lower bound.

Under PT, the monetary policy rule is replaced by a modified Taylor rule that can be written as

$$i_t^d = r_t + \pi^* + \rho_p\left(p_t - p_t^*\right) + \rho_y^* y_t, \tag{5}$$

where p_t is the price level (in logs) and where π_t^* is the projected path of the price-level target (also in logs). The price-level target path evolves according to

$$p_t^* = p_{t-1}^* + \pi^*,$$

where once again π^* is trend inflation. This allows for a price-level target that is not necessarily constant. The realized nominal interest rate is still given by (4). The main distinguishing feature between IT and PT is whether or not unexpected shocks that affect the inflation rate are corrected in the long run or not.

Under PT, it will be convenient to consider the following transformed version of the model, which introduces the deviation between the price level and its target path as an extra state variable. The Phillips curve (1) can be rewritten as follows:

$$\left(p_t - p_{t-1}\right) = \left(1-\beta\right)\pi^* + \beta \mathrm{E}_t\left(p_{t+1} - p_t\right) + \varphi y_t$$

[8] We could add a stochastic error term to this equation, but the results of our paper would not be affected.

$$\Rightarrow (p_t - p_t^*) - \left(p_{t-1} - p_{t-1}^*\right) + \left(p_t^* - p_{t-1}^*\right)$$

$$= (1-\beta)\,\pi^* + \beta \mathrm{E}_t \left(p_{t+1} - p_{t+1}^*\right) - \beta\,(p_t - p_t^*) + \beta\left(p_{t+1}^* - p_t^*\right) + \varphi y_t.$$

Since $\left(p_t^* - p_{t-1}^*\right) = \left(p_{t+1}^* - p_t^*\right) = \pi^*$, we get

$$(p_t - p_t^*) - \left(p_{t-1} - p_{t-1}^*\right) = \beta \mathrm{E}_t \left(p_{t+1} - p_{t+1}^*\right) - \beta\,(p_t - p_t^*) + \varphi y_t. \quad (6)$$

The New Keynesian IS curve (2) becomes

$$y_t = \mathrm{E}_t y_{t+1} - \frac{1}{\gamma}\left(i_t - \mathrm{E}_t\left(p_{t+1} - p_{t+1}^*\right) + (p_t - p_t^*) - \pi^* - r_t\right) + v_t. \quad (7)$$

The other equations of the model require no transformations.

We also assume that the natural real rate of interest follows the stochastic process given by

$$r_t \sim N\left(r, \sigma_r^2\right). \quad (8)$$

3 Deterministic Steady States

3.1 Deterministic Steady State under IT

Even before the 2007 financial crisis some researchers questioned the stability properties of the IT framework. BSU (2001, 2001b) showed that IT regimes must theoretically have two steady states under perfect foresight. There is one equilibrium in which inflation is equal to its target. The other equilibrium is a “liquidity-trap” equilibrium with the nominal interest rate stuck at or near its lower bound and characterized by deflation. The existence of multiple steady states is due to the kink in the Taylor rule at the zero lower bound.

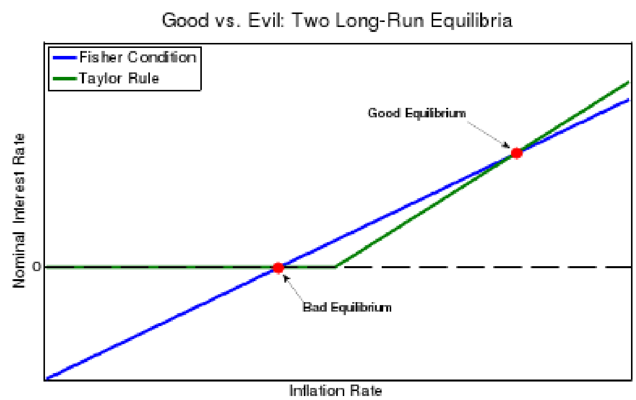
Figure 1 (from Mendes, 2011) illustrates their argument. The Fisher relation gives a linear relation (with a slope of one) between steady-state inflation and the nominal interest rate. The Taylor rule together with the zero lower bound imply a kinked relation between inflation and the nominal interest rate. The positively-sloped segment of this curve has a slope greater than one if the Taylor principle is satisfied (if $\rho_\pi > 1$). This means that there must be two points of intersection between the two curves and hence two steady states. The steady state with a zero nominal interest rate has the property that $\pi = -r$. This satisfies the Friedman rule, but the equilibrium is “bad” in this context because it implies a negative output gap which is potentially quite large depending on the value of the model’s parameters.

We show in Appendix A that there are exactly two deterministic steady states for our model under IT, in line with BSU. The Friedman rule is satisfied in the liquidity-trap steady, and the output gap is given by

$$y = -\frac{(1-\beta)}{\varphi}\left(r + \pi^*\right).$$

The output gap is negative and potentially quite large if φ is small (if inflation is insensitive to the output gap). This will be the case with large nominal price rigidities (if firms adjust their prices infrequently) or with large real rigidities

Figure 1:



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From Mendes (2011)

(firms' optimal reset prices are not very sensitive to the output gap).[9]

The result for the deterministic steady state holds under perfect foresight. Evans, Guse and Honkapohja (2008) showed the possibility of large shocks like the one that initiated the Great Recession leading to deflationary spirals in environments with expectations formed using an adaptive learning rule. Bullard (2010) argued that the low-inflation equilibrium trap was empirically relevant for Japan in the first decade of the century and could easily have become relevant for the U.S. in the wake of the 2008 financial crisis.

4 Deterministic Steady State under PT

We show in Appendix B that there can only be one deterministic steady state in which the deviation of the price level from its target path is constant, that is

$$\left(p_{t+1} - p^*_{t+1}\right) = \left(p_t - p^*_t\right) = \left(p_{t-1} - p^*_{t-1}\right) \equiv p^d$$

This equilibrium must have the property that the deviation of the price level p^d must be equal to zero, which also implies a zero output gap. This result would seem to imply that the economy cannot remain stuck indefinitely at the zero lower bound.

There is also a "quasi-steady-state equilibrium", equivalent to the liquidity-trap

[9]The target rate of inflation π^* is exogenous here and is implicitly taken to be positive. This begs the question of the choice of the optimal target rate of inflation, which we do not consider here. Coibion, Gorodnichenko and Wieland (2012) address this question using the New Keynesian model as a framework, and show that the optimal rate of inflation is positive, precisely in order to reduce the probability that the nominal interest rate hits its zero lower bound.

equilibrium in the IT case. We characterize this quasi steady state in Appendix B, starting from the assumption that the realized interest rate is at its lower bound. All of the model's variables are constant in this quasi steady state except for the price-level gap $(p_t - p_t^*)$, which decreases at a rate equal to $-(r + \pi^*)$.

Since the price-level gap is not constant, the central bank's desired interest rate is also decreasing over time. However, there is no feedback from this gap to the rest of the model as long as the realized nominal interest rate is stuck at zero. There is no mechanism to pry the economy away from this low-inflation quasi steady state.[10]

This quasi steady state has undesirable properties, like the low-inflation steady state under IT. To maintain a zero nominal interest rate and a constant rate of inflation, it must be the case that

$$y = -\frac{(1-\beta)}{\varphi}(r + \pi^*).$$

The output gap is negative, and once again potentially large if φ is small.

We show in the next section that as long as agents expect that a shock will eventually push the economy away from the liquidity-trap quasi steady state, so that the unconditional expectation of the realized nominal interest rate is bounded above zero, the only possible stochastic steady state is one in which the inflation rate is equal on average to its target rate.

[10]We refer to this as a quasi steady state since we have introduced the price-level gap $(p_t - p_t^*)$ as a state variable. Honkapohja and Mitra (2014) call it a steady state. The difference is semantic: in their model, it is also the case that the price level gap increases without bound but has no feedback effect on other variables.

5 Stochastic Steady States

5.1 Stochastic Steady State under IT

This case has been covered in detail by Mendes (2011). He shows that there can be either two, one, or zero stochastic steady states in a model like the one developed here. The two-steady-state case is similar to the deterministic case and holds when the volatility of stochastic shocks to the real rate of interest is sufficiently low. If the volatility of the real interest rate is sufficiently high, the expected nominal interest rate for any given rate of inflation increases. This follows because the nominal interest rate i_t has a distribution that is left-truncated at zero. An increase in the variance of the innovation to r_t given by σ_r^2 pushes out the right tail of the distribution of i_t, thereby increasing its unconditional mean. With a high enough expected policy rate, the expected policy rule is everywhere above the expected Fisher condition and there is no steady state at which the two curves intersect.[11]

5.2 Stochastic Steady State under PT

In Appendix C, we show that if there is a stochastic steady state under the PT regime, it is unique. Furthermore, it must have the characteristic that the gap between the price level and the desired price-level path is constant. This immediately implies that the inflation rate is on average equal to the target rate of

[11] The case of one stochastic steady state is the razor's edge case where the volatility of stochastic shocks is just high enough that the expected policy function lies above the expected Fisher condition and intersects it at one point.

inflation. We also show that the low-inflation deterministic quasi steady state of the previous subsection does not exist when we introduce stochastic shocks.

The stochastic steady state under PT has several interesting properties. As noted in the previous paragraph, the unconditional expectation of the inflation rate is equal to target inflation. This means that there is no inflationary or deflationary bias under PT. The expected value of the output gap is zero.[12] The expected value of the realized interest rate is just the unconditional mean of the real interest rate plus the target inflation rate. There is a wedge between the unconditional expectation of the desired interest rate and the realized interest rate. This follows from equation (4) which implies that the realized interest rate is a (left) truncated variable compared to the desired interest rate. Taking unconditional expectations of the modified Taylor rule leads immediately to the following expression for the relation between the wedge and the expected price-level gap.

$$\mathrm{E}p^d = -\frac{1}{\rho_p}\left(\mathrm{E}i - \mathrm{E}i^d\right) < 0.$$

The unconditional expectation of the price level gap is negative, and depends inversely on ρ_p, the parameter that determines how strongly the central bank reacts to the price-level gap. Under pure price-level targeting with $\rho_p \to \infty$, the expected price level gap and the wedge between the realized and desired interest rate disappear. This means that as the central bank reacts more and more strongly against deviations of the price level from its target path, the probability of hitting

[12] This would hold only approximately in a model in which the New Keynesian Phillips curve and the dynamic IS equation were not linearized.

the lower bound goes goes to zero.

The economic intuition for these results is straightforward. Given the modified Taylor rule, the central bank has a commitment to restore the price level to its target path after any shock. Even if the economy is at in an equilibrium in which the zero lower bound on the policy rate is binding, agents expect that sooner or later a positive shock will occur that will move the economy away from the lower bound. Then, along the transition path back to the target price-level path, inflation will be higher than the target rate π^*. Averaging over periods where the economy is at the zero bound and periods where it is not, inflation is equal to the target rate.

We also show that the desired nominal interest rate, while lower on average than the realized nominal interest rate because of the zero-bound problem, is arbitrarily close to the realized interest rate on average as the central bank reacts more and more strongly to price-level deviations, that is to say for large values of ρ_p.

6 Determinacy in Parameter Space

We use a simple Monte Carlo approach based on Ratto (2008) to analyze the stability of the model in parameter space. Dittmar and Gavin (2005) already explored, in the context of a standard New Keynesian model, regions of the parameter space under IT and PT and concluded that the model's dynamics were determinate for a wider range of parameter values under PT than under IT.

The advantage of the methodology proposed by Ratto (2008) is that it explores the parameter space in a systematic way, and uncovers the parameters that are most important for determining stability versus instability and indeterminacy.

The model's high-inflation steady state coincides under IT and PT. The solution implies $\pi = \pi^*$, $y = 0$, $i = r + \pi^*$, and (under PT) $(p_t - p_t^*) = 0$. The steady-state solutions are independent of parameter values, so we can approximate the model's dynamics around the same point independent of parameter values.

The linearized dynamics of the model under IT and PT are given in Appendix D. Under IT, the model has no predetermined state variables. Saddlepoint stability requires that there be two unstable roots corresponding to the two forward-looking variables. Under PT, there is one predetermined variable and there are two non-predetermined variables. For saddlepoint stability, we require one stable and two unstable eigenvalues.

We allow ρ_y^*, ρ_p and φ to vary, drawing from a uniform distribution. Table 1 below specifies the supports of the distributions for these parameters. When the support for a parameter is degenerate, its value is held fixed across replications. We considered that the key parameters for the Monte Carlo exercise were ρ_p, ρ_y and φ. The ρ_p parameter is the equivalent of the ρ_π parameter under IT, the sensitivity of the policy rate to variations in inflation. This parameter has been the focus of analyses of the Taylor principle in the literature. An increase in ρ_y can offset a decrease in the value of ρ_p in circumstances where the output gap and inflation are both either above or below their target. If the policy rate is not reacting strongly enough to inflation to modify the real interest rate in the required direction, changes in the output gap will move the policy rate in the required direction to stabilize both the output gap and inflation. The φ parameter is an important part of the transmission of monetary policy to inflation, since a change in the policy rate affects inflation via its impact on aggregate demand and the output gap.[13]

We drew 50,000 sets of parameter values. For each set we checked the values of the eigenvalues evaluated at the steady state. We find no cases where saddlepoint stability was violated. This indicates that as long as the central bank responds positively to deviations of the price level from the target path and to the output gap, the model has determinate dynamics. The intuition for this result is clear. In response to a deviation of the price level from its target path, the response of the policy rate is cumulative. If inflation rises even slightly above the long-run target value, the price level will gradually deviate more and more from the target path. The interest rate response eventually becomes strong enough to move the ex ante real interest rate in the right direction.

6.1 Sunspots

Having ruled out the possibility of indeterminacy under PT, we can also rule out bubbles or sunspot solutions. Karnizova (2010) shows in a standard New Keynesian model very similar to the one used here that sunspot terms are

[13] The parameters that are held constant here, β and γ, do not substantially affect the values of the stable and unstable roots of the dynamic model.

Table 1: Parameter value ranges for Monte Carlo

parameter	min	max
ρ_y	0.0	1.0
ρ_p	0.0	2.0
φ	0.01	0.5
β	0.995	0.995
γ	1.0	1.0

irrelevant to the model's solution whenever the model's equilibrium is determinate[14] She concludes (page 9) that "non-fundamental expectation revisions caused by sunspots can arise only under indeterminacy." A similar result was shown by Farmer (1999, chapter 10) for a simple real business cycle model with possibly increasing returns to scale: only the model with increasing returns and indeterminacy admits solutions in which sunspots are relevant. Since indeterminacy can be ruled out under PT, sunspots or bubbles can be ruled out as well. This is yet another advantage of PT over IT as a monetary policy framework.

7 Global Stability

In engineering and physics, where dynamic models typically contain only predetermined state variables (the dynamics do not result partly from the presence of forward-looking economic agents), sophisticated techniques such as Lyapunov equations are available to check the global stability of nonlinear

[14] under IT the model is determinate with "active" monetary.

dynamical systems. The development of techniques for analyzing global stability are less developed for nonlinear economic models with forward-looking or non-predetermined state variables.

To analyze global stability (which in the presence of forward-looking costate variables means saddlepoint stability) in our model, we adapted the technique proposed by Brunner and Strulik (2002), who proposed the technique as a method of solving nonlinear rational expectations or perfect foresight models. We take advantage of the fact that it has only one predetermined state variable. The convergent arm of the saddle is a line in our three-dimensional parameter space.

Our model is "globally saddlepoint stable" in the following sense. Solving the model backwards from any terminal values for the model's state variable and its two costate variables, we can verify that the paths converge backwards towards the convergent arm of the saddle. For a given value of the model's predetermined variable $\left(p_{t-1} - p^*_{t-1}\right)$ sufficiently far away from its steady-state value of zero, the values of costate variables of $\left(p_{t-1} - p^*_{t-1}\right)/y_t$ and $\left(p_{t-1} - p^*_{t-1}\right)/\left(p_t - p^*_t\right)$ are arbitrarily close together.

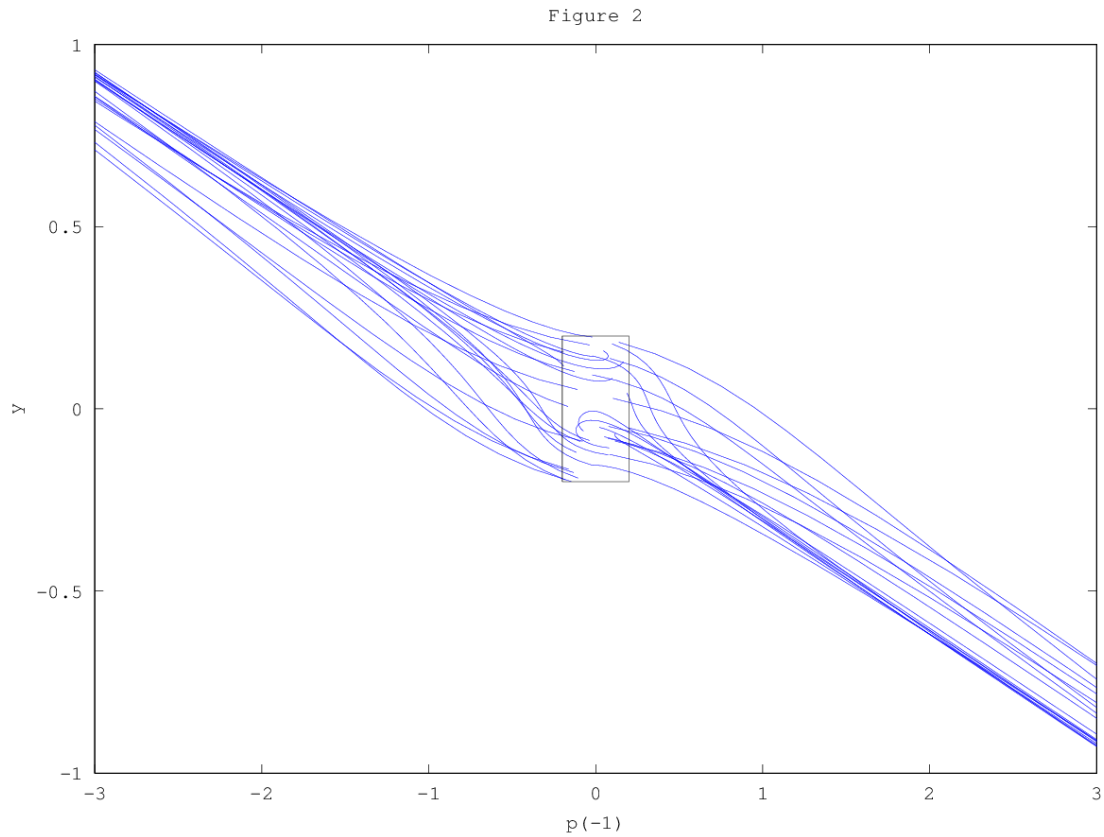
We verified this using a simple recursive algorithm for solving the model backwards from given terminal conditions and picking those terminal conditions using a Monte Carlo technique. More details on the solution algorithm are given in Appendix E. We drew values from a joint uniform distribution for $y_T \in [-0.2, 0.2]$, $\left(p_T - p^*_{T-1}\right) \in [-0.2, 0.2]$ and $\left(p_{T-1} - p^*_{T-1}\right) \in [-0.2, 0.2]$ to use as terminal values for our backward simulations. This allows for an output gap of up to 20% and a price-level gap of up to 20%, ranges which easily encompass all empirically relevant states. We then iterated the model backwards until the absolute value of p_{T-1} was well outside the range of the terminal values. We used $\left| \left(p_{t-1} - p^*_{t-1}\right) \right| \geq 10,000$ as a criterion. With one predetermined state variable (p_{t-1}) and two non-predetermined variables (y_t and p_t), the stable arm of the economy's saddlepoint is one-dimensional. We checked that $\frac{\left(p_{t-1}-p^*_{t-1}\right)}{y_t}$ and $\frac{\left(p_{t-1}-p^*_{t-1}\right)}{(p_t-p^*_t)}$ were sufficiently close to each other for all terminal values (the starting values for the backward simulations). This was in fact the case, with $\frac{\left(p_{t-1}-p^*_{t-1}\right)}{y_t} = -3.2802$ and $\frac{\left(p_{t-1}-p^*_{t-1}\right)}{(p_t-p^*_t)} = 1.2085$.

Figure 2 shows some sample paths (forty draws from the joint distribution for the terminal values) for the three dynamic variables of the model projected onto the two-dimensional plane in y_t and $\left(p_{t-1} - p^*_{t-1}\right)$. All of the illustrated paths converge backwards towards the convergent arm of the saddle.[15]

This is a constructive proof of the global (saddlepoint) stability of the economy under PT. It shows that the initial conditions for y_t and $(p_t - p^*_t)$ to be on the convergent arm of the saddle are unique.[16]

[15] Some of the paths intersect. This is because the figure is a projection of three-dimensional dynamics onto a two-dimensional plane.

[16] Our global stability results hold under rational expectations. In a deterministic model with learning, Honkapohja and Mitra (2014) demonstrate local stability of price-level targeting near the targeted steady state with small price adjustment costs and numerically verify local stability for a wide range of parameter values.



8 Conclusions

We have shown that in a canonical New-Keynesian model PT eliminates the problem of multiple steady-states that is inherent under an IT regime. We have shown that the economy's dynamics are determinate in the region of its unique steady state for a very wide range of parameter values: the central bank does not need to respond strongly to deviations of the price level from its target path for determinacy, and there is no equivalent of the Taylor principle that the central bank must satisfy under PT. Over a wide range of states, the economy's dynamics are saddlepoint stable, showing that PT leads to global determinacy. Our results underscore a possible advantage of PT compared to IT that has escaped the attention of the literature. This should encourage central banks to give PT a closer look as a possible monetary policy framework, especially in order to avoid deflationary spirals when policy rates are near the zero lower bound.

Appendix

A Deterministic Steady State under IT

We show the existence of precisely two deterministic steady states in this case. Dropping time subscripts from the equations of the model gives

$$\pi = (1-\beta)\pi^* + \beta\pi + \varphi y, \tag{9}$$

$$i = r + \pi, \tag{10}$$

$$i^d = r + \pi^* + \rho_\pi \pi - \rho_\pi \pi^* + \rho_y y, \tag{11}$$

$$i = \max\left(0, i^d\right). \tag{12}$$

There are two possible cases, $i = i^d > 0$ and $i = 0$. First consider the case with a positive nominal interest rate in the steady state. Equations (10) and (11) together imply that

$$\pi = \pi^* + \rho_\pi \pi - \rho_\pi \pi^* + \rho_y y$$

$$\Rightarrow (1 - \rho_\pi)(\pi - \pi^*) = \rho_y y,$$

while equation (9) implies

$$(1 - \beta)(\pi - \pi^*) = \varphi y.$$

We have two linear equations in two unknowns, the first of which has a positive slope and the second of which has a negative slope. The unique solution is $y = 0$ and $\pi = \pi^*$. This is the steady state where inflation is equal to target inflation and the output gap is zero.

Now consider the case where $i = 0$. Equation (11) now just gives the level of the desired interest rate in the deterministic steady state, which must be negative. The Fisher relation (10) gives

$$\pi = -r.$$

Substituting into (9) and solving gives the following unique solution for the output gap:

$$y = -\frac{(1 - \beta)}{\varphi}(r + \pi^*).$$

This is the low-inflation steady state. It is clearly an undesirable steady state given the model. The inflation rate is equal to the negative of the real interest rate, which satisfies the Friedman rule, but the economy is stuck with a negative income gap which is potentially quite large. It would be theoretically possible to eliminate the negative output gap by setting $\pi^* = -r$. This is just the Friedman rule. As is well known, it also has the advantage of equating the real rates of return on money and short-term bonds, leading to a socially-optimal level of real money balances (of course money demand does not explicitly enter our model). While this works in a deterministic setting, Mendes (2011) shows that it leads to non-existence of the steady state when stochastic shocks to the real interest rate are added to the model.

B Deterministic Steady State under PT

B.1 True Steady State

First, consider a true steady state in which all of the model's state variables are constant, in particular

$$\left(p_{t+1} - p^*_{t+1}\right) = \left(p_t - p^*_t\right) = \left(p_{t-1} - p^*_{t-1}\right) \equiv p^d,$$

where p^d is the deviation of the price level from its target path. The value of p^d is possibly different from zero, but in fact it is easy to show that this cannot be the case. The transformed version of the New Keynesian Phillips curve (2) immediately gives

$$0 = \varphi y \Rightarrow y = 0.$$

Substituting into the New Keynesian IS curve (7), we immediately get

$$i = r + \pi^*.$$

The only true steady state has an output gap of zero and a positive nominal interest rate. The modified Taylor rule (5) then implies that

$$(p_t - p_t^*) = p^d = 0.$$

The price level follows its target path in the steady state.

B.2 Quasi Steady State

If we start by simply assuming $i = 0$, we can back out the following solutions for the other variables of the model in the long run. The untransformed version of the New Keynesian IS curve (2) then immediately implies that

$$\pi = -r.$$

Once again, we have the Friedman rule, but this will again imply a negative output gap in the steady state. Substituting in the transformed version of the New Keynesian IS curve (7) gives

$$\left(p_{t+1} - p_{t+1}^*\right) - (p_t - p_t^*) = -r - \pi^*,$$

which implies (using the transformed version of the New Keynesian Phillips curve) that

$$y = -\frac{(1-\beta)}{\varphi}\left(r + \pi^{*}\right).$$

We get the same solution for inflation, the output gap, and the nominal interest rate as in the liquidity-trap steady state under IT.

The solution is a “quasi” steady state because one of the model’s state variables, the gap between the price level and its target path, is not at rest. With a negative rate of inflation, this gap decreases without bound, and the central bank’s desired interest rate also decreases without bound. However, since the constraint of the zero bound is binding in this equilibrium, there is no feedback from the price-level gap to the rest of the model.

C Stochastic Steady State under PT

Mendes (2011) gives and exhaustive treatment of the stochastic steady state under IT. He shows that the liquidity-trap equilibrium under IT involves an expected nominal interest rate that remains constant and is superior to the expected desired interest rate. The lower bound makes the realized interest rate a left-truncated normal random variable, whose expectation depends positively on the variance of the underlying shocks in the model.

Here, we consider the existence of either a steady state in which the unconditional expectations of all of the model’s state variables are constant, or a quasi steady state in which all variables have constant unconditional means except for possibly the gap between the price level and its desired path and the desired interest rate. In the quasi steady state, the unconditional expectation of the inflation rate is constant so that

$$\mathrm{E}\left(p_t^d - p_{t-1}^d\right) \equiv \mathrm{E}\Delta p_t^d \quad \forall t$$

$$\equiv \mathrm{E}\Delta p^d$$

is constant. This implies that $\mathrm{E}p_t^d$ is a deterministic function of time.

Dropping time subscripts, and taking unconditional expectations of variables, we get

$$\mathrm{E}\Delta p^d = \beta \mathrm{E}\Delta p^d + \varphi \mathrm{E}y,$$

$$\mathrm{E}y = \mathrm{E}y - \frac{1}{\gamma}\left(Ei - \mathrm{E}\Delta p^d - \pi^* - r\right),$$

$$\mathrm{E}i_t^d = r + \pi^* + \rho_p \mathrm{E}p_t^d + \rho_y^* \mathrm{E}y,$$

$$\mathrm{E}i = \mathrm{E}\max\left(0, i_t^d\right).$$

We immediately arrive at a contradiction. The expectation of the realized interest rate depends on the expectation of a nonlinear function of a variable that is not constant, so it cannot be constant. Therefore, there is no steady state that satisfies the criterion that variables other than the gap between the price level and its target path (and the desired interest rate) are constant.

So if a stochastic steady state with these properties exists, it must be the case that the unconditional expectation of the deviation of the price level is constant. We must have

$$\mathrm{E}\Delta p^d = 0.$$

From the first equation we must have $\mathrm{E}y = 0$. If the stochastic equilibrium exists, it must be the case that the unconditional expectation of the output gap is zero. The intuition for this result is straightforward. With any expected inflation rate that is different from π^*, the expected price-level gap must be changing over time. The modified Taylor rule then implies that the unconditional expectation of the desired interest rate must be changing over time, which implies that the unconditional expectation of the realized nominal interest rate cannot be constant.

We then get, from the New Keynesian IS curve, that

$$\mathrm{E}i = r + \pi^*.$$

Substituting into the modified Taylor rule gives

$$\mathrm{E}i^d = \mathrm{E}i + \rho_p \mathrm{E}p^d.$$

If the shocks of the model (the shock to the real interest rate and the shock to the modified Taylor rule itself) are normally distributed, the unconditional distributions of the variables in the model must be normal, and the realized real interest rate is a truncated normal distribution. It is left-truncated, so it must be the case that

$$\mathrm{E}i > \mathrm{E}i^d.$$

We have

$$\mathrm{E}p^d = -\frac{1}{\rho_p}\left(\mathrm{E}i - \mathrm{E}i^d\right) < 0.$$

On average, there will be a non-zero price-level gap. Its expected value is negative and depends on the strength with which the central bank varies its desired interest rate in response to the price-level gap. Under pure price-level gap targeting, as $\rho_p \to \infty$, the expected price-level gap tends to zero. The interpretation of this is clear. If the central bank reacts strongly against price-level deviations from the desired price-level path, the zero bound will rarely be binding and the desired nominal interest rate will be close, on average, to the realized nominal interest rate.

D Linearised Dynamics

D.1 Dynamics under IT

In approximating the dynamics of the model around the steady state where $\pi = \pi^*$, we ignore the zero bound constraint on the nominal interest rate. The system can easily be reduced to the following two-equation system:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{\gamma} & 1 \end{bmatrix} \begin{bmatrix} \mathrm{E}_t\left(\pi_{t+1} - \pi^*\right) \\ \mathrm{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\varphi}{\beta} \\ \frac{\rho_\pi}{\gamma} & 1 + \frac{\rho_y}{\gamma} \end{bmatrix} \begin{bmatrix} \left(\pi_t - \pi^*\right) \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} v_t.$$

D.2 Dynamics under PT

Once again ignoring the zero lower bound constraint on the interest rate, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{E}_t \left(p_{t+1} - p^*_{t+1}\right) \\ \mathrm{E}_t y_{t+1} \\ (p_t - p^*_t) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1+\beta}{\beta} & -\frac{\varphi}{\beta} & -\frac{1}{\beta} \\ \frac{1+\rho_p}{\gamma} & 1+\frac{\rho^*_y}{\gamma} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} (p_t - p^*_t) \\ y_t \\ \left(p_{t-1} - p^*_{t-1}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} v_t.$$

The last equation in the system is a simple identity that equates the first lag of the forward-looking variable $\mathrm{E}_t \left(p_{t+1} - p^*_{t+1}\right)$ with the lead of the predetermined state variable $(p_t - p^*_t)$.

E Backward Solution

We ignore stochastic shocks, setting $r_t = r$ and $v_t = 0$, and drop the expectations operator. We start off with arbitrary values for $\left(p_{t+1} - p^*_{t+1}\right)$, y_{t+1} and $(p_t - p^*_t)$. Then, inverting the equation system from the previous subsection, we get

$$\begin{bmatrix} (p_t - p^*_t) \\ y_t \\ \left(p_{t-1} - p^*_{t-1}\right) \end{bmatrix} = \begin{bmatrix} \frac{1+\beta}{\beta} & -\frac{\varphi}{\beta} & -\frac{1}{\beta} \\ \frac{1+\rho_p}{\gamma} & 1+\frac{\rho^*_y}{\gamma} & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{\gamma} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \left(p_{t+1} - p^*_{t+1}\right) \\ y_{t+1} \\ (p_t - p^*_t) \end{bmatrix}.$$

Given this solution, we check the policy rule to make sure that the zero lower bound constraint does not bite, using

$$i_t^d = r + \pi^* + \rho_p (p_t - p_t^*) + \rho_y^* y_t.$$

We ignore stochastic shocks to the natural real rate of interest. If the zero bound constraint is binding, we substitute $i_t = 0$ in the New Keynesian IS equation and solve the following dynamical system in place of the original one:

$$\begin{bmatrix} (p_t - p_t^*) \\ y_t \\ \left(p_{t-1} - p_{t-1}^*\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varphi & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\gamma} & 1 & -\frac{1}{\gamma} \\ -\beta & 0 & (1+\beta) \end{bmatrix} \begin{bmatrix} \left(p_{t+1} - p_{t+1}^*\right) \\ y_{t+1} \\ (p_t - p_t^*) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varphi & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{\gamma} \\ 0 \end{bmatrix} (r + \pi^*) .$$

We solve the model backwards using arbitrary end values for $\left(p_{t+1} - p_{t+1}^*\right)$, y_{t+1} and $(p_t - p_t^*)$. We stop the iterations when the absolute value of y_t is such that we are sufficiently far away from the model's steady-state equilibrium. Each backward solution path should be arbitrarily close to the convergent arm of the saddle.

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